

ON SUBREGULARITY PROPERTIES OF SET-VALUED MAPPINGS. APPLICATIONS TO SOLID VECTOR OPTIMIZATION

by

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Abstract: In this work we classify the at-point regularities of set-valued mappings into two categories and then we analyze their relationship through several implications and examples. After this theoretical tour, we use the subregularity properties to deduce implicit theorems for set-valued maps. Finally, we present some applications to the study of multicriteria optimization problems.

Keywords: set-valued maps · at-point regularity · around-point regularity · implicit multifunction theorems · solid vector optimization

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1 Introduction

This work is a natural continuation of the recent papers [19] and [16]. We mention that in [19] several results concerning the behavior of the implicit solution mappings associated to parametric variational systems are given under around-point regularity assumptions for the initial mappings, while in [16] the same concepts are employed in order to get necessary optimality conditions for multicriteria optimization. However, let us remark that, on one hand, the calmness property replaces the Aubin continuity in many recent works in literature in order to get well-posedness results for parametric systems (see, for instance, [2], [9], [25] and the references therein) and, on the other hand, in certain situations, metric subregularity of the constraint system is enough for deriving necessary optimality conditions for generalized mathematical programs (see [35], [23]).

In this perspective, we revisit some results in [19], [16], trying to replace, where possible, the around-point regularities of set-valued mappings by weaker concepts of at-point regularity. First of all, we classify the regularity notions at the reference point into two categories, which we call first and, respectively, second type at-point regularity. Following the pattern stated for regularity around the reference point, for both these types we consider the corresponding triads of openness, metric regularity and Lipschitzness. For the first type we have the at-point openness at linear rate ([34]), the metric hemiregularity ([2]) and the pseudocalmness ([19]). For the second type, which seems to be of greater interest compared to the first one, up to now there exist only two concepts: the

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calmness and the metric subregularity (see [15, Section 3H]). Our first aim is to complete the triad by an appropriate equivalent openness notion (which we call the linear pseudo-openness), giving us the possibility to have a deeper insight on the results involving the second type of regularity at the reference point. After that, we establish the relationship between all these notions, with a special emphasis on the case of linear bounded operators.

With all these facts in mind, we are able to discuss our main results, which are divided into three themes. The first one concerns implicit multifunction type theorems using at-point regularity of the second type, showing the technical advantages of the linear pseudo-openness concept. This is in the line with the remark that, in general, the linear openness notions are technically easier to deal with in the proofs, while the equivalent metric regularities are more useful in applications (see [12], [11]). The second theme is devoted to the analysis of the local-sum stability, a notion recently introduced in [19] in relation with the conservation of the Aubin property for sum-type set-valued maps. In this work we study the natural link between this sort of stability and the conservation of calmness at summation, and we emphasize by examples other situations where the sum multifunction is calm, in the absence of this property for the component mappings. The third theme is dedicated to the study of variational systems in the context of the second type at-point regularity, taking advantage of the technical analysis developed before. Firstly, we derive a result concerning the metric subregularity of the implicit mappings associated to parametric variational systems, and we provide an example which shows that the sole around-point regularity assumption used to this aim cannot be dropped. Secondly, we provide a fixed-point result for the parametric case of composition of two set-valued mappings, which can be seen in relation to some recent metric extensions of the Lyusternik-Graves theorem (see [13], [14], [20], [21]). Then we use the above mentioned result to deduce, under additional appropriate conditions, the calmness of the solution mappings of the parametric systems.

In the final section of the paper we employ the second type of at-point regularity in the study of solid set-valued optimization problems. First of all, using the metric subregularity of the constraint system, we combine a Clarke type penalization technique and a scalarization result in order to reduce the problem of getting necessary optimality conditions for weak Pareto minimizers to the one of finding local minimizers for a scalar function. After that, we use some ideas from [16] in order to deduce sufficient conditions for the needed metric subregularity of the constraint system, following the error bounds approach and using the Mordukhovich generalized differentiation objects for the formulation of regularity by means of dual objects. Putting all the previous facts together, we finally get the expected necessary optimality conditions in terms of Mordukhovich differentiation, by expressing some generalized Lagrange multipliers rules in the normal form for the proposed optimization problem.

2 Preliminaries

In what follows, we suppose that the involved spaces are metric spaces, unless otherwise stated. In this setting, $B(x, r)$ and $D(x, r)$ denote the open and the closed ball with center x and radius r , respectively. On a product space we take the additive metric. If $x \in X$ and $A \subset X$, one defines the distance from x to A as $d(x, A) := \inf\{d(x, a) \mid a \in A\}$. As usual, we use the convention $d(x, \emptyset) = \infty$. The excess from a set A to a set B is defined as $e(A, B) := \sup\{d(a, B) \mid a \in A\}$. For a non-empty set $A \subset X$ we put $\text{cl} A$ for its topological closure. One says that a set A is locally closed around $x \in A$ if there exists $r > 0$ such that $A \cap D(x, r)$ is closed.

Let $F : X \rightrightarrows Y$ be a multifunction. The domain and the graph of F are denoted respectively by

$\text{Dom } F := \{x \in X \mid F(x) \neq \emptyset\}$ and $\text{Gr } F := \{(x, y) \in X \times Y \mid y \in F(x)\}$. If $A \subset X$ then $F(A) := \bigcup_{x \in A} F(x)$. The inverse set-valued map of F is $F^{-1} : Y \rightrightarrows X$ given by $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$.

We denote by P the metric space of parameters. For a (parametric) multifunction $F : X \times P \rightrightarrows Y$, we use the notations: $F_p(\cdot) := F(\cdot, p)$ and $F_x(\cdot) := F(x, \cdot)$.

We divide the reminder of this section into three different subsections, each one being dedicated to a certain type of regularity in set-valued setting.

2.1 Around-point regularity

We recall now the concepts of openness at linear rate, metric regularity and Aubin property of a multifunction around the reference point. Generally, when one speaks about regularity for a set-valued map, one refers to these concepts.

Definition 2.1 *Let $L > 0$, $F : X \rightrightarrows Y$ be a multifunction and $(\bar{x}, \bar{y}) \in \text{Gr } F$.*

(i) F is said to be open at linear rate L , or L -open around (\bar{x}, \bar{y}) , if there exist a positive number $\varepsilon > 0$ and two neighborhoods $U \in \mathcal{V}(\bar{x})$, $V \in \mathcal{V}(\bar{y})$ such that, for every $\rho \in (0, \varepsilon)$ and every $(x, y) \in \text{Gr } F \cap [U \times V]$,

$$B(y, \rho L) \subset F(B(x, \rho)). \quad (2.1)$$

The supremum of $L > 0$ over all the combinations (L, U, V, ε) for which (2.1) holds is denoted by $\text{lop } F(\bar{x}, \bar{y})$ and is called the exact linear openness bound, or the exact covering bound of F around (\bar{x}, \bar{y}) .

(ii) F is said to have the Aubin property around (\bar{x}, \bar{y}) with constant L if there exist two neighborhoods $U \in \mathcal{V}(\bar{x})$, $V \in \mathcal{V}(\bar{y})$ such that, for every $x, u \in U$,

$$e(F(x) \cap V, F(u)) \leq Ld(x, u). \quad (2.2)$$

The infimum of $L > 0$ over all the combinations (L, U, V) for which (2.2) holds is denoted by $\text{lip } F(\bar{x}, \bar{y})$ and is called the exact Lipschitz bound of F around (\bar{x}, \bar{y}) .

(iii) F is said to be metrically regular around (\bar{x}, \bar{y}) with constant L if there exist two neighborhoods $U \in \mathcal{V}(\bar{x})$, $V \in \mathcal{V}(\bar{y})$ such that, for every $(x, y) \in U \times V$,

$$d(x, F^{-1}(y)) \leq Ld(y, F(x)). \quad (2.3)$$

The infimum of $L > 0$ over all the combinations (L, U, V) for which (2.3) holds is denoted by $\text{reg } F(\bar{x}, \bar{y})$ and is called the exact regularity bound of F around (\bar{x}, \bar{y}) .

The next proposition contains the well-known links between the notions presented above. See [33], [26], [15] for more details about its proof and for historical facts.

Proposition 2.2 *Let $F : X \rightrightarrows Y$ be a multifunction and $(\bar{x}, \bar{y}) \in \text{Gr } F$. Then F is open at linear rate around (\bar{x}, \bar{y}) iff F^{-1} has the Aubin property around (\bar{y}, \bar{x}) iff F is metrically regular around (\bar{x}, \bar{y}) . Moreover, in every of the previous situations,*

$$(\text{lop } F(\bar{x}, \bar{y}))^{-1} = \text{lip } F^{-1}(\bar{y}, \bar{x}) = \text{reg } F(\bar{x}, \bar{y}).$$

All the regularity concepts given before have parametric counterparts, which we present next.

Definition 2.3 Let $L > 0$, $F : X \rightrightarrows Y$ be a multifunction and $((\bar{x}, \bar{p}), \bar{y}) \in \text{Gr } F$.

(i) F is said to be open at linear rate L , or L -open, with respect to x uniformly in p around $((\bar{x}, \bar{p}), \bar{y})$ if there exist a positive number $\varepsilon > 0$ and some neighborhoods $U \in \mathcal{V}(\bar{x})$, $V \in \mathcal{V}(\bar{p})$, $W \in \mathcal{V}(\bar{y})$ such that, for every $\rho \in (0, \varepsilon)$, every $p \in V$ and every $(x, y) \in \text{Gr } F_p \cap [U \times W]$,

$$B(y, \rho L) \subset F_p(B(x, \rho)). \quad (2.4)$$

The supremum of $L > 0$ over all the combinations $(L, U, V, W, \varepsilon)$ for which (2.4) holds is denoted by $\widehat{\text{lop}}_x F((\bar{x}, \bar{p}), \bar{y})$ and is called the exact linear openness bound, or the exact covering bound of F in x around $((\bar{x}, \bar{p}), \bar{y})$.

(ii) F is said to have the Aubin property with respect to x uniformly in p around $((\bar{x}, \bar{p}), \bar{y})$ with constant L if there exist some neighborhoods $U \in \mathcal{V}(\bar{x})$, $V \in \mathcal{V}(\bar{p})$, $W \in \mathcal{V}(\bar{y})$ such that, for every $x, u \in U$ and every $p \in V$,

$$e(F_p(x) \cap W, F_p(u)) \leq Ld(x, u). \quad (2.5)$$

The infimum of $L > 0$ over all the combinations (L, U, V, W) for which (2.5) holds is denoted by $\widehat{\text{lip}}_x F((\bar{x}, \bar{p}), \bar{y})$ and is called the exact Lipschitz bound of F in x around $((\bar{x}, \bar{p}), \bar{y})$.

(iii) F is said to be metrically regular with respect to x uniformly in p around $((\bar{x}, \bar{p}), \bar{y})$ with constant L if there exist some neighborhoods $U \in \mathcal{V}(\bar{x})$, $V \in \mathcal{V}(\bar{p})$, $W \in \mathcal{V}(\bar{y})$ such that, for every $(x, p, y) \in U \times V \times W$,

$$d(x, F_p^{-1}(y)) \leq Ld(y, F_p(x)). \quad (2.6)$$

The infimum of $L > 0$ over all the combinations (L, U, V, W) for which (2.6) holds is denoted by $\widehat{\text{reg}}_x F((\bar{x}, \bar{p}), \bar{y})$ and is called the exact regularity bound of F in x around $((\bar{x}, \bar{p}), \bar{y})$.

The corresponding notions with respect to p uniformly in x can be written similarly.

In the sequel, we emphasize the fact that the corresponding "at-point" properties could be separated into two different categories, which for the sake of clarity we present as type I and type II.

2.2 At-point regularity: type I

The first type of at-point regularity contains the linear openness at point, the pseudocalmness and the metric hemiregularity, as follows.

Definition 2.4 Let $L > 0$, $F : X \rightrightarrows Y$ be a multifunction and $(\bar{x}, \bar{y}) \in \text{Gr } F$.

(i) F is said to be open at linear rate L , or L -open at (\bar{x}, \bar{y}) if there exists a positive number $\varepsilon > 0$ such that, for every $\rho \in (0, \varepsilon)$,

$$B(\bar{y}, \rho L) \subset F(B(\bar{x}, \rho)). \quad (2.7)$$

The supremum of $L > 0$ over all the combinations (L, ε) for which (2.7) holds is denoted by $\text{plop } F(\bar{x}, \bar{y})$ and is called the exact punctual linear openness bound of F at (\bar{x}, \bar{y}) .

(ii) F is said to be pseudocalm with constant L , or L -pseudocalm at (\bar{x}, \bar{y}) , if there exists a neighborhood $U \in \mathcal{V}(\bar{x})$ such that, for every $x \in U$,

$$d(\bar{y}, F(x)) \leq Ld(x, \bar{x}). \quad (2.8)$$

The infimum of $L > 0$ over all the combinations (L, U) for which (2.8) holds is denoted by $\text{psdclm } F(\bar{x}, \bar{y})$ and is called the exact bound of pseudocalmness for F at (\bar{x}, \bar{y}) .

(iii) F is said to be *metrically hemiregular with constant L* , or *L -metrically hemiregular at (\bar{x}, \bar{y})* , if there exists a neighborhood $V \in \mathcal{V}(\bar{y})$ such that, for every $y \in V$,

$$d(\bar{x}, F^{-1}(y)) \leq Ld(y, \bar{y}). \quad (2.9)$$

The infimum of $L > 0$ over all the combinations (L, V) for which (2.9) holds is denoted by $\text{hemreg } F(\bar{x}, \bar{y})$ and is called the *exact hemiregularity bound of F at (\bar{x}, \bar{y})* .

As in the case of around-point regularity, some equivalences between these notions hold. For the proof, see, for instance, [19, Proposition 2.4].

Proposition 2.5 *Let $L > 0$, $F : X \rightrightarrows Y$ and $(\bar{x}, \bar{y}) \in \text{Gr } F$. Then F is L -open at (\bar{x}, \bar{y}) iff F^{-1} is L^{-1} -psdclm at (\bar{y}, \bar{x}) iff F is L^{-1} -metrically hemiregular at (\bar{x}, \bar{y}) . Moreover, in every of the previous situations,*

$$(\text{plop } F(\bar{x}, \bar{y}))^{-1} = \text{psdclm } F^{-1}(\bar{y}, \bar{x}) = \text{hemreg } F(\bar{x}, \bar{y}).$$

Let us mention that partial corresponding variants could be done as in the previous subsection.

2.3 At-point regularity: type II

The second type of at-point regularity contains the calmness, the metric subregularity, and a notion we introduce here under the name of linear pseudo-openness. This novelty serves to complete the regularity triad in this case. Moreover, it proves to be useful in the attempt of getting implicit multifunction results under weaker assumptions, and this is the main aim of this work.

Definition 2.6 *Let $L > 0$, $F : X \rightrightarrows Y$ be a multifunction and $(\bar{x}, \bar{y}) \in \text{Gr } F$.*

(i) F is said to be *linearly pseudo-open with modulus L* , or *L -pseudo-open at (\bar{x}, \bar{y})* if there exist $U \in \mathcal{V}(\bar{x})$ and $\varepsilon > 0$ such that for every $\rho \in (0, \varepsilon)$ and for every $x \in U \cap F^{-1}(B(\bar{y}, L\rho))$,

$$\bar{y} \in F(B(x, \rho)). \quad (2.10)$$

The supremum of $L > 0$ over all the combinations (L, U, ε) for which (2.10) holds is denoted by $\text{lpo } F(\bar{x}, \bar{y})$ and is called the *exact linear pseudo-openness bound of F at (\bar{x}, \bar{y})* .

(ii) F is said to be *calm with constant L* , or *L -calm at (\bar{x}, \bar{y})* , if there exists some neighborhoods $U \in \mathcal{V}(\bar{x})$, $V \in \mathcal{V}(\bar{y})$ such that, for every $x \in U$,

$$e(F(x) \cap V, F(\bar{x})) \leq Ld(x, \bar{x}). \quad (2.11)$$

The infimum of $L > 0$ over all the combinations (L, U, V) for which (2.11) holds is denoted by $\text{clm } F(\bar{x}, \bar{y})$ and is called the *exact bound of calmness for F at (\bar{x}, \bar{y})* .

(iii) F is said to be *metrically subregular with constant L* , or *L -metrically subregular at (\bar{x}, \bar{y})* if there exists a neighborhood $U \in \mathcal{V}(\bar{x})$ such that, for every $x \in U$,

$$d(x, F^{-1}(\bar{y})) \leq Ld(\bar{y}, F(x)). \quad (2.12)$$

The infimum of $L > 0$ over all the combinations (L, U) for which (2.9) holds is denoted by $\text{subreg } F(\bar{x}, \bar{y})$ and is called the *exact subregularity bound of F at (\bar{x}, \bar{y})* .

Remark that the properties of calmness and metrically subregularity are well known. The first concept of Definition 2.6 which we introduce here plays the same role in this triplet as the openness in the above well-known ones. The exact meaning of this assertion is given in the next result.

Proposition 2.7 *Let $F : X \rightrightarrows Y$ and $(\bar{x}, \bar{y}) \in \text{Gr } F$. Then F is linearly pseudo-open at (\bar{x}, \bar{y}) iff F^{-1} is calm at (\bar{y}, \bar{x}) iff F is metrically subregular at (\bar{x}, \bar{y}) . Moreover, in every of the previous situations,*

$$(\text{lpo } F(\bar{x}, \bar{y}))^{-1} = \text{clm } F^{-1}(\bar{y}, \bar{x}) = \text{subreg } F(\bar{x}, \bar{y}).$$

Proof. The equivalence between the calmness of F^{-1} and the metric subregularity of F , as well as the relation between the corresponding regularity moduli are well-known (see, for instance, [15, Section 3H]).

Let us prove the equivalence between the linear pseudo-openness of F and the calmness of F^{-1} . Suppose first that F is linearly pseudo-open at (\bar{x}, \bar{y}) with modulus $L > 0$. Then there exist $U \in \mathcal{V}(\bar{x})$ and $\varepsilon > 0$ such that, for every $\rho \in (0, \varepsilon)$ and every $x \in U \cap F^{-1}(B(\bar{y}, L\rho))$, $\bar{y} \in F(B(x, \rho))$. Consider $\varepsilon' := L\varepsilon$, $V := B(\bar{y}, \varepsilon')$, and take $y \in V$ and $x \in F^{-1}(y) \cap U$. Without loosing the generality, suppose that $y \neq \bar{y}$, because otherwise the desired relation trivially holds. Then, there exists $\rho \in (0, \varepsilon)$ such that $d(y, \bar{y}) = L\rho < L\varepsilon$. Take $\tau > 0$ arbitrary small such that $\rho' := \rho + \tau < \varepsilon$. Consequently, $x \in U \cap F^{-1}(B(\bar{y}, L\rho'))$, hence $\bar{y} \in F(B(x, \rho'))$, or there exists $z \in B(x, \rho')$ such that $z \in F(\bar{y})$. But this means that

$$d(x, F(\bar{y})) \leq d(x, z) \leq \rho + \tau = L^{-1}d(y, \bar{y}) + \tau.$$

Because x was arbitrarily taken from $F^{-1}(y) \cap U$, from the previous relation one deduces that

$$e(F^{-1}(y) \cap U, F(\bar{y})) \leq L^{-1}d(y, \bar{y}) + \tau$$

for every $\tau > 0$ arbitrary small, so making $\tau \rightarrow 0$ one deduces that F^{-1} is calm at (\bar{y}, \bar{x}) with modulus L^{-1} .

Suppose now that F^{-1} is calm at (\bar{y}, \bar{x}) with modulus L^{-1} , so there exists $U \in \mathcal{V}(\bar{x})$ and $V \in \mathcal{V}(\bar{y})$ such that, for every $y \in V$,

$$e(F^{-1}(y) \cap U, F(\bar{y})) \leq L^{-1}d(y, \bar{y}).$$

We will prove that F is linearly pseudo-open at (\bar{x}, \bar{y}) with modulus smaller, but arbitrarily close to L . Choose $\xi > 0$ such that $L - \xi > 0$, take $\varepsilon > 0$ such that $B(\bar{y}, L\varepsilon) \subset V$, fix arbitrary $\rho \in (0, \varepsilon)$ and $x \in U \cap F^{-1}(B(\bar{y}, (L - \xi)\rho))$. Then there exists $y \in B(\bar{y}, (L - \xi)\rho) \subset B(\bar{y}, L\rho) \subset B(\bar{y}, L\varepsilon) \subset V$ such that $x \in F^{-1}(y)$. Therefore, $d(x, F^{-1}(\bar{y})) \leq L^{-1}d(y, \bar{y})$, so one can find $z \in F^{-1}(\bar{y})$ such that

$$d(x, z) < L^{-1}d(y, \bar{y}) + L^{-1}\rho\xi < L^{-1}(L - \xi)\rho + L^{-1}\rho\xi = \rho.$$

In conclusion, there exists $z \in B(x, \rho)$ such that $\bar{y} \in F(z)$, i.e. the conclusion. \square

Remark 2.8 *Notice that, in our notation, we obviously have the following characterizations:*

- (i) *F is calm at (\bar{x}, \bar{y}) with constant $L > 0$ if and only if there exist $U \in \mathcal{V}(\bar{x})$ and $V \in \mathcal{V}(\bar{y})$ such that for every $x \in U$ and $y \in V$ with $d(y, F(\bar{x})) > Ld(x, \bar{x})$, we have $y \notin F(x)$;*
- (ii) *F is metrically subregular at (\bar{x}, \bar{y}) with constant $L > 0$ if and only if there exist $U \in \mathcal{V}(\bar{x})$ and $V \in \mathcal{V}(\bar{y})$ such that for every $x \in U$ and $y \in V$ with $d(x, F^{-1}(\bar{y})) > Ld(y, \bar{y})$, we have $y \notin F(x)$.*

Even if one can define all the corresponding partial notions to the concepts in Definition 2.6, we restrict ourselves to the case of calmness, because this is exactly what we use in the next sections. More precisely, F is said to be calm with respect to x uniformly in p at $((\bar{x}, \bar{p}), \bar{y})$ with constant $L > 0$ if there exist some neighborhoods $U \in \mathcal{V}(\bar{x})$, $V \in \mathcal{V}(\bar{p})$, $W \in \mathcal{V}(\bar{y})$ such that, for every $x \in U$ and every $p \in V$,

$$e(F_p(x) \cap W, F_p(\bar{x})) \leq Ld(x, \bar{x}). \quad (2.13)$$

The infimum of $L > 0$ over all the combinations (L, U, V, W) for which (2.13) holds is denoted by $\widehat{\text{clm}}_x F((\bar{x}, \bar{p}), \bar{y})$ and is called the exact calmness bound of F in x at $((\bar{x}, \bar{p}), \bar{y})$.

We close this section by some comments concerning the comparison of the three sets of concepts we have listed above. It is well known (and easy to see) that around-point regularity implies both types of at-point regularity while the converse implications obviously fail.

A remarkable situations where several interesting assertions could be additionally established is that of linear continuous operators, i.e. the case where F is replaced by $A \in \mathcal{L}(X, Y)$, where $\mathcal{L}(X, Y)$ denotes the normed vector space of linear bounded operators acting between X and Y . A consequence of Banach Open Principle is that A is open at linear rate around a (every) (x, Ax) if and only if A is surjective. Moreover, following [2, Proposition 5.2], this is further equivalent to the hemiregularity of A at a (every) (x, Ax) . Since for a linear continuous operator all the regularity moduli are not depending on the reference point we can remove it from their notations and, under surjectivity of A , one has

$$\text{psdclm } A^{-1} = \text{lip } A^{-1} = \text{hemreg } A = \text{reg } A = (\text{plop } A)^{-1} = (\text{lop } A)^{-1} = \|(A^*)^{-1}\|, \quad (2.14)$$

where $A^* \in \mathcal{L}(Y^*, X^*)$ denotes the adjoint operator of A .

For the case of second type at-point regularities the situation changes significantly. First of all, if X and Y are finite dimensional, then every $A \in \mathcal{L}(X, Y)$ is metrically subregular at every point of its graph. To see this, observe that if X, Y are finite dimensional and $A \in \mathcal{L}(X, Y)$, then $\text{Im } A$ is isomorphic with X_2 , where by X_2 we denote the algebraic complement of $\text{Ker } A$ in X . One has to prove that there is $L > 0$ such that, for every $x \in X$,

$$\inf\{\|x - u\| \mid u \in \text{Ker } A\} \leq L \|Ax\|. \quad (2.15)$$

Observe first that if $x \in \text{Ker } A$, then the above inequality trivially holds. Define next the isomorphism $A_1 : X_2 \rightarrow \text{Im } A$, given by $A_1 x := Ax$ for every $x \in X_2$, and apply the Banach Open Principle for A_1 to deduce that A_1 is open. Equivalently, there exists $L > 0$ such that, for every $y \in \text{Im } A$, there exists $x \in X_2$ such that $y = A_1 x$ and $\|x\| \leq L \|y\|$. Take arbitrary $x \in X_2$. Then it uniquely corresponds to $y = A_1 x = Ax \in \text{Im } A$, hence

$$\|x\| \leq L \|Ax\|.$$

Finally, for an arbitrary $x \in X$, we decompose it as $x = x' + x''$, with $x' \in \text{Ker } A$ and $x'' \in X_2$, and we have

$$\inf\{\|x - u\| \mid u \in \text{Ker } A\} \leq \|x - x'\| = \|x''\| \leq L \|Ax''\| = L \|Ax\|,$$

which ends the proof. Note that an alternative proof can be given as follows: one knows that the distance in the left-hand side of (2.15) is attained and can be written as $\langle x, x^* \rangle$ where x^* belongs to orthogonal subspace of $\text{Ker } A$ and its operatorial norm is 1 (see [36, Theorem 3.8.4 (vii)]). By

means of Farkas Lemma ([32, Corrolary 22.3.1]), x^* can be expressed as a linear combination of the scalar linear components mappings of A and then the thesis is proved by a manipulation of some easy inequalities. For other details concerning these aspects, see the comments in [15, Section 3H]. The previous discussion means that any linear operator on finite dimensional spaces which is not surjective is metrically subregular, but fails to be metrically regular. Therefore, even for linear operators, metric subregularity does not imply metric regularity.

On infinite dimensional spaces, there exist linear bounded operators which fail to be metrically subregular. For instance, consider the spaces m and l^2 of bounded and respectively square-summable sequences of real numbers with their usual norms and $T : m \rightarrow l^2$ with $T((x_n)) = (n^{-1}x_n)$ for every sequence $(x_n)_{n \in \mathbb{N} \setminus \{0\}}$. It is easy to show that T is well defined, linear, continuous ($\|T\| = \sqrt{6^{-1}}\pi$) and injective. Nevertheless, supposing that T would be metrically subregular at $(0,0)$, then should exist $L > 0$ s.t. for every $(x_n) \in m$,

$$\|(x_n)\| \leq L \|T(x_n)\|.$$

Taking, for every natural $k \neq 0$, $(x_n^k)_n$ as the sequence having all the components zero except that on k -th place which is 1, then the above relation reads as $1 \leq Lk^{-1}$ for every $k \in \mathbb{N} \setminus \{0\}$. This is a contradiction which can be eliminated only if T is not metrically subregular at $(0,0)$.

On the other hand, on Banach spaces, if A is injective and its image is closed, then this is equivalent to the following property:

$$\exists M > 0 \text{ s.t. } \forall x \in X, \|x\| \leq M \|Ax\|. \quad (2.16)$$

It is easy to observe that this last relation easily leads to the linear pseudo-openness at every point of the graph. Indeed, it is enough to see that in these assumptions one can write for $(\bar{x}, \bar{y}) = (0,0)$ and every $x \in X$

$$d(x, A^{-1}(0)) \leq \|x - 0\| \leq M \|Ax\| = Md(0, Ax),$$

which ensures the metric subregularity of A . With this remark, we infer that in the above example, the metric subregularity fails because the image of T is not closed in l^2 .

Let us observe now that if A is open (hence surjective) with constant $L > 0$ then obviously A is pseudo-open with modulus L and $\text{lpo } A \leq \text{lpo } A$.

The opposite inequality is also true. Indeed, if A is pseudo-open then we find $\varepsilon > 0$ and a neighborhood $U \in \mathcal{V}(0)$ such that for every $\rho \in (0, \varepsilon)$ and for every $x \in U \cap A^{-1}(B(0, L\rho))$ we have that $0 \in A(B(x, \rho))$. Starting with $y \in B(0, L\rho)$, since A is surjective, then there exists $x \in X$ with $Ax = y$, and taking into account that U is a neighborhood of 0 then we find $\lambda \geq 1$ and $x' \in U$ such that $x = \lambda x'$. Now, we have that $\|Ax'\| \leq \lambda^{-1}\rho L$, therefore $0 \in A(B(x', \lambda^{-1}\rho))$ because $x' \in U \cap A^{-1}(B(0, \lambda^{-1}\rho L))$. From this and from linearity of A we obtain that $0 \in A(B(x, \rho))$, which shows that A is open at linear rate L and, taking into account (2.14), $\text{lpo } A \leq \text{plo } A = \text{lpo } A$. In conclusion, under surjectivity of A , one can add to (2.14) the following chain of equalities:

$$\text{clm } A^{-1} = \text{lip } A^{-1} = \text{subreg } A = \text{reg } A = (\text{lpo } A)^{-1} = (\text{lop } A)^{-1} = \|(A^*)^{-1}\|.$$

3 Main results

This section is divided into three subsections, as follows:

- the first one consists of an implicit multifunction type theorem displaying at-point regularity of the second type;
- the second one gives further insights on recently introduced notion of local sum-stability of two multifunctions; more precisely, we investigate the relation with the calmness of the sum of two mappings both through theoretical results and examples;
- the third one concerns the study of parametric variational systems in the context of the second type at-point regularity.

3.1 At-point regularity for implicit multifunctions

In this subsection we obtain a result concerning at-point regularity of the second type for a generalized implicit set-valued map.

To this aim, we consider a setting coming from the study of parametric variational systems. Given a multifunction $H : X \times P \rightrightarrows Y$, define the implicit mapping $S : P \rightrightarrows X$ by

$$S(p) := \{x \in X \mid 0 \in H(x, p)\}. \quad (3.1)$$

The study of the well-posedness properties of S , including the so-called Robinson regularity, was a constant issue of operations research in the last four decades (see [15] for a extended discussion and historical comments).

The next result is in the line of [19, Theorem 3.6], but for the newly introduced concept of linear pseudo-openness instead of the classical genuine linear openness around the reference point, and for calmness instead of Aubin property.

Theorem 3.1 *Let X, P be metric spaces, Y be a normed vector space, $H : X \times P \rightrightarrows Y$ be a set-valued map and $(\bar{x}, \bar{p}, 0) \in \text{Gr } H$. Denote by $H_p(\cdot) := H(\cdot, p)$, $H_x(\cdot) := H(x, \cdot)$.*

(i) If $H_{\bar{p}}$ is linearly pseudo-open with modulus $c > 0$ at $(\bar{x}, 0)$, then there exist $\alpha, \beta > 0$ such that, for every $x \in B(\bar{x}, \alpha)$,

$$d(x, S(\bar{p})) \leq c^{-1} d(0, H(x, \bar{p}) \cap B(0, \beta)). \quad (3.2)$$

If, moreover, H is calm with respect to p uniformly in x at $((\bar{x}, \bar{p}), 0)$, then S is calm at (\bar{p}, \bar{x}) and

$$\text{clm } S(\bar{p}, \bar{x}) \leq c^{-1} \widehat{\text{clm}}_p H((\bar{x}, \bar{p}), 0). \quad (3.3)$$

(ii) If $H_{\bar{x}}$ is linearly pseudo-open with modulus $c > 0$ at $(\bar{p}, 0)$, then there exist $\gamma, \delta > 0$ such that, for every $p \in B(\bar{p}, \gamma)$,

$$d(p, S^{-1}(\bar{x})) \leq c^{-1} d(0, H(\bar{x}, p) \cap B(0, \delta)). \quad (3.4)$$

If, moreover, H is calm with respect to x uniformly in p at $((\bar{x}, \bar{p}), 0)$, then S is metrically subregular at (\bar{p}, \bar{x}) and

$$\text{subreg } S(\bar{p}, \bar{x}) \leq c^{-1} \widehat{\text{clm}}_x H((\bar{x}, \bar{p}), 0). \quad (3.5)$$

Proof. Observe first that is sufficient to prove the (i) item, because the second item follows symmetrically, using $T := S^{-1}$ instead of S , and taking into account Proposition 2.7.

Let us prove the first part. We know that there exist $r, \varepsilon > 0$ such that, for every $\rho \in (0, \varepsilon)$ and every $x \in B(\bar{x}, r) \cap H_{\bar{p}}(B(0, c\rho))$, one has $0 \in H_{\bar{p}}(B(x, \rho))$.

Consider $\rho \in (0, \varepsilon)$, $\alpha := r$, $\beta := c\rho$, and take $x \in B(\bar{x}, \alpha)$. If $H(x, \bar{p}) \cap B(0, \beta) = \emptyset$, the relation (3.2) automatically holds. Suppose next that $H(x, \bar{p}) \cap B(0, \beta) \neq \emptyset$. If $0 \in H(x, \bar{p}) \cap B(0, \beta)$, then, again, (3.2) trivially holds. Consider now the case $0 \notin H(x, \bar{p}) \cap B(0, \beta)$. Then for every $\xi > 0$, one can find $y_\xi \in H(x, \bar{p}) \cap B(0, \beta)$ such that

$$\|y_\xi\| < d(0, H(x, \bar{p}) \cap B(0, \beta)) + \xi.$$

Then

$$0 \in B(y_\xi, d(0, H(x, \bar{p}) \cap B(0, \beta)) + \xi).$$

Because $d(0, H(x, \bar{p}) \cap B(0, \beta)) < \beta = c\rho$, one can select $\xi > 0$ sufficiently small such that $d(0, H(x, \bar{p}) \cap B(0, \beta)) + \xi < c\rho$. Define $\rho_0 := c^{-1}(d(0, H(x, \bar{p}) \cap B(0, \beta)) + \xi) < \rho < \varepsilon$. Observe now that $y_\xi \in H(x, \bar{p}) \cap B(0, c\rho_0)$, which means that $x \in H_{\bar{p}}^{-1}(y_\xi) \subset H_{\bar{p}}^{-1}(B(0, c\rho_0))$. Consequently, $x \in B(\bar{x}, r) \cap H_{\bar{p}}^{-1}(B(0, c\rho_0))$, so using the assumption made we deduce that $0 \in H_{\bar{p}}(B(x, \rho_0))$. Equivalently, there exists $\tilde{x} \in B(x, \rho_0)$ such that $\tilde{x} \in S(\bar{p})$. In conclusion,

$$d(x, S(\bar{p})) \leq d(x, \tilde{x}) < \rho_0 = c^{-1}d(0, H(x, \bar{p}) \cap B(0, \beta)) + c^{-1}\xi.$$

Making $\xi \rightarrow 0$, one obtains (3.2).

Suppose next that H is calm with respect to p uniformly in x at $(\bar{x}, \bar{p}, 0)$, so there exist $s, t, l > 0$ such that $ls < c\rho$, and for every $(x, p) \in B(\bar{x}, s) \times B(\bar{p}, s)$,

$$e(H(x, p) \cap B(0, t), H(x, \bar{p})) \leq ld(p, \bar{p}).$$

Consider $a := \min\{\alpha, s\}$, and take $p \in B(\bar{p}, s)$, $x \in S(p) \cap B(\bar{x}, a)$. Then $0 \in H(x, p) \cap B(0, t)$, so $d(0, H(x, \bar{p})) \leq ld(p, \bar{p})$. For every $\tau > 0$ sufficiently small such that $ls + \tau < c\rho$, there is $y_\tau \in H(x, \bar{p})$ such that

$$\|y_\tau\| < ld(p, \bar{p}) + \tau < c\rho.$$

In conclusion, $y_\tau \in H(x, \bar{p}) \cap B(0, c\rho)$, which means, using (3.2), that

$$d(x, S(\bar{p})) \leq c^{-1}d(0, H(x, \bar{p}) \cap B(0, c\rho)) \leq c^{-1}\|y_\tau\| < c^{-1}ld(p, \bar{p}) + c^{-1}\tau.$$

Making $\tau \rightarrow 0$ in the relation $d(x, S(\bar{p})) < c^{-1}ld(p, \bar{p}) + c^{-1}\tau$, and using the arbitrariness of $x \in S(p) \cap B(\bar{x}, a)$, one deduces the calmness of S at (\bar{p}, \bar{x}) . Also, the relation between the associated moduli of calmness easily follows. \square

Notice that Theorem 3.1 (i) could be compared with [9, Theorem 3.1], where the same conclusion is obtained under somehow stronger assumptions in terms of coderivatives and using a closed-graph assumption for H . Moreover, our result could be deduced using even weaker concepts of openness, but we preferred the actual form for consistency with results in the sequel.

The next examples emphasize the fact that in Theorem 3.1 the converses do not hold.

Example 3.2 Consider the multifunctions $H : \mathbb{R} \times \mathbb{R} \rightrightarrows \mathbb{R}$, given by

$$H(x, p) := \begin{cases} \{0\}, & \text{if } |x| \geq |p| \\ \{\sqrt{|p|}\}, & \text{if } |x| < |p|. \end{cases}$$

Then $S : \mathbb{R} \rightrightarrows \mathbb{R}$ is given by $S(p) = \mathbb{R} \setminus (-|p|, |p|)$. Take $(\bar{x}, \bar{p}) = (0, 0)$. It is easy to prove that S is metrically subregular at $(0, 0)$ with constant 1 and calm at $(0, 0)$ with constant 1.

On the other hand, $H_{\bar{x}}(p) = \{\sqrt{|p|}\}$ for any p and $H_{\bar{p}}(x) = \{0\}$ for any x , whence $H_{\bar{x}}$ is linearly pseudo-open at $(\bar{p}, 0)$ and $H_{\bar{p}}$ is linearly pseudo-open at $(\bar{x}, 0)$.

Finally, observe that H is not calm with respect to x uniformly in p at $((0, 0), 0)$ and H is not calm with respect to p uniformly in x at $((0, 0), 0)$. If we suppose, by way of contradiction, that H is calm with respect to x uniformly in p at $((0, 0), 0)$ then there exist $L > 0$, $U \in \mathcal{V}(0)$, $V \in \mathcal{V}(0)$ and $W \in \mathcal{V}(0)$ such that for every $x \in U$ and $p \in V$

$$e(H(x, p) \cap W, H(0, p)) \leq Ld(x, 0),$$

hence we find $n_0 \in \mathbb{N}$ such that $\sqrt{\frac{1}{n}} \leq L\frac{1}{n}$ for every $n \geq n_0$, which is not true. Therefore, H is not calm with respect to x uniformly in p at $((0, 0), 0)$. Similarly, taking $p = \frac{1}{n}$ and $x = \frac{1}{n^2}$ we deduce that H is not calm with respect to p uniformly in x at $((0, 0), 0)$.

Example 3.3 Consider the multifunction $H : \mathbb{R} \times \mathbb{R} \rightrightarrows \mathbb{R}$, given by

$$H(x, p) := \begin{cases} [0, 1], & \text{if } |x| \geq |p| \\ (0, 1], & \text{if } |x| < |p|. \end{cases}$$

Then S is the same as in the previous example, whence it is metrically subregular at $(\bar{x}, \bar{p}) = (0, 0)$ but $H_{\bar{x}}$ is not linearly pseudo-open at $(0, 0)$.

Since $H_0(p)$ is $[0, 1]$ if $|p| = 0$ and $(0, 1]$ if $|p| \neq 0$, H_0 is not linearly pseudo-open at $(0, 0)$ because if we suppose the opposite, taking $x = \frac{1}{n}$, $y = \frac{1}{n^2}$ and $\rho = \frac{1}{Ln^2}$ we find $n_0 \in \mathbb{N}$ such that $\frac{1}{n} < \frac{1}{Ln^2}$ for every $n \geq n_0$, which is not true. In contrast, H is calm with respect to x uniformly in p at $((0, 0), 0)$.

Remark 3.4 In both items of Theorem 3.1, one can replace the calmness assumptions by somehow weaker (but more technical) conditions, as follows:

(i) Suppose that (3.2) holds. If there exists $M > 0$ such that for every $x \in B(\bar{x}, \alpha)$ and for every p in a neighborhood of \bar{p} s.t. $d(p, \bar{p}) < Md(0, H(x, \bar{p}) \cap B(0, \beta))$ it follows that $0 \notin H(x, p)$, then S is calm at (\bar{p}, \bar{x}) with constant $c^{-1}M^{-1}$;

(ii) Suppose that (3.4) holds. If there exists $M > 0$ such that for every $p \in B(\bar{p}, \gamma)$ and for every x in a neighborhood of \bar{x} s.t. $d(x, \bar{x}) < Md(0, H(\bar{x}, p) \cap B(0, \delta))$ it follows that $0 \notin H(x, p)$, then S is metrically subregular at (\bar{p}, \bar{x}) with constant $c^{-1}M^{-1}$.

Indeed, for (i), take $x \in B(\bar{x}, \alpha)$ and p close to \bar{p} s.t. $d(x, S(\bar{p})) > c^{-1}M^{-1}d(p, \bar{p})$. Then

$$c^{-1}d(0, H(x, \bar{p}) \cap B(0, \beta)) \geq d(x, S(\bar{p})) > c^{-1}M^{-1}d(p, \bar{p}),$$

therefore $0 \notin H(x, p)$, i.e. $x \notin S(p)$. Taking into account Remark 2.8, S is calm with constant $c^{-1}M^{-1}$ at (\bar{p}, \bar{x}) . The second item follows symmetrically.

Observe that there are situations where this remark applies while Theorem 3.1 does not. This is shown in the example below.

Consider the multifunction $H : \mathbb{R} \times \mathbb{R} \rightrightarrows \mathbb{R}$, given by

$$H(x, p) := \begin{cases} \left\{ \frac{|x|}{|p|} \right\}, & \text{if } p \neq 0 \text{ and } x \neq 0, \\ \{0\}, & \text{if } p = 0 \text{ and } x \neq 0, \\ \{|p|\}, & \text{if } x = 0 \end{cases}$$

and take $\bar{x} = \bar{p} = 0$. Then $H_{\bar{x}}(p) = \{|p|\}$ is linearly pseudo-open with modulus $c = 1$ at $(\bar{p}, 0)$ and there exist $\gamma = \delta = 1$ such that for every $p \in B(\bar{p}, \gamma)$,

$$d(p, S^{-1}(\bar{x})) = |p| = d(0, H(\bar{x}, p) \cap B(0, \delta)).$$

For $M = 1$, if $d(x, \bar{x}) < Md(0, H(\bar{x}, p) \cap B(0, \delta))$ then $|x| < |p| < 1$, whence $0 \notin H(x, p)$ because $0 \in H(x, p)$ if and only if $p = 0$. Then S is metrically subregular at $(0, 0)$ according to the remark. For this case we cannot apply Theorem 3.1 because H is not calm with respect to x uniformly in p at $((0, 0), 0)$. To observe this, take $x = p = \frac{1}{n}$ and, arguing by contradiction, one obtains $1 - \frac{1}{n} \leq L \frac{1}{n}$ for every $n \in \mathbb{N}$ large enough, which is not possible.

3.2 Local sum-stability

This subsection revisits the concept of local-sum stability introduced in [19, Section 4]. Originally, this was used in relation with the Aubin property of the sum-multifunction, while here we follow a similar procedure, but for calmness. The notion itself reads as follows.

Definition 3.5 ([19, Definition 4.2]) *Let $F : X \rightrightarrows Y$, $G : X \rightrightarrows Y$ be two multifunctions and $(\bar{x}, \bar{y}, \bar{z}) \in X \times Y \times Y$ such that $\bar{y} \in F(\bar{x})$, $\bar{z} \in G(\bar{x})$. We say that the multifunction (F, G) is locally sum-stable around $(\bar{x}, \bar{y}, \bar{z})$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $x \in B(\bar{x}, \delta)$ and every $w \in (F + G)(x) \cap B(\bar{y} + \bar{z}, \delta)$, there exist $y \in F(x) \cap B(\bar{y}, \varepsilon)$ and $z \in G(x) \cap B(\bar{z}, \varepsilon)$ such that $w = y + z$.*

Besides the initial results involving this notion (see [19, Section 4]), it was recently used and studied in relation with the metric regularity of the sum of multifunctions in [28].

We begin our analysis announced before by considering an example which shows that the calmness property is not stable under summation (see, for more details, [19, Example 4.8], where a similar example is given to prove that the Aubin property does not hold for the sum of multimapings). Take the multifunctions $F, G : \mathbb{R} \rightrightarrows \mathbb{R}$, given by

$$F(x) := \begin{cases} [0, 1] \cup \{2\}, & \text{if } x \in \mathbb{R} \setminus \{0\} \\ [0, 1], & \text{if } x = 0 \end{cases}$$

and by $G(x) := [0, 1]$ for every $x \in \mathbb{R}$, which are calm at $(0, 1)$. Then the multifunction $F + G : \mathbb{R} \rightrightarrows \mathbb{R}$, given by

$$(F + G)(x) = \begin{cases} [0, 3], & \text{if } x \in \mathbb{R} \setminus \{0\} \\ [0, 2], & \text{if } x = 0, \end{cases}$$

is not calm at $(0, 2)$.

The next lemma, whose proof is straightforward, shows that, as in the case of Aubin property, the local sum-stability is the missing ingredient in order to get the conservation of the calmness property at summation.

Lemma 3.6 *Let $F : X \rightrightarrows Y$, $G : X \rightrightarrows Y$ be two multifunctions. Suppose that F is calm at $(\bar{x}, \bar{y}) \in \text{Gr } F$, that G is calm at $(\bar{x}, \bar{z}) \in \text{Gr } G$, and that (F, G) is locally sum-stable around $(\bar{x}, \bar{y}, \bar{z})$. Then the multifunction $F + G$ is calm at $(\bar{x}, \bar{y} + \bar{z})$. Moreover, the following relation holds true*

$$\text{clm}(F + G)(\bar{x}, \bar{y} + \bar{z}) \leq \text{clm } F(\bar{x}, \bar{y}) + \text{clm } G(\bar{x}, \bar{z}). \quad (3.6)$$

Proof. Use the calmness properties of F and G , to get $\alpha, l, k > 0$ such that, for every $x \in B(\bar{x}, \alpha)$,

$$e(F(x) \cap B(\bar{y}, \alpha), F(\bar{x})) \leq ld(x, \bar{x}), \quad (3.7)$$

$$e(G(x) \cap B(\bar{z}, \alpha), G(\bar{x})) \leq kd(x, \bar{x}). \quad (3.8)$$

Using the locally sum-stability for $\varepsilon := \alpha > 0$, one can find $\delta \in (0, \alpha)$ such that, for every $x \in B(\bar{x}, \delta)$ and every $w \in (F + G)(x) \cap B(\bar{y} + \bar{z}, \delta)$, there exist $y \in F(x) \cap B(\bar{y}, \alpha)$ and $z \in G(x) \cap B(\bar{z}, \alpha)$ such that $w = y + z$. Consequently, using (3.7) and (3.8),

$$d(w, (F + G)(\bar{x})) \leq d(y, F(\bar{x})) + d(z, G(\bar{x})) \leq (l + k)d(x, \bar{x}).$$

The relation (3.6) easily follows. \square

Another remark is that, unsurprisingly, the calmness of the sum-multifunction can be obtained in various situations, without the calmness of the component multifunctions and in the absence of any kind of local sum-stability.

The next proposition, whose proof is omitted being again straightforward, uses another sort of stability for the component mappings in proving the calmness of the sum.

Proposition 3.7 *Let $F : X \rightrightarrows Y$, $G : X \rightrightarrows Y$ be two multifunctions and $(\bar{x}, \bar{y}) \in \text{Gr } F$, $(\bar{x}, \bar{z}) \in \text{Gr } G$. Suppose that there exist $L_F, L_G > 0$ and $U \in \mathcal{V}(\bar{x})$, $V \in \mathcal{V}(\bar{y} + \bar{z})$ such that, for every $x \in U$ and $w \in V$ with $(x, w) \in \text{Gr}(F + G)$, we can find $y \in F(x)$ with $d(y, F(\bar{x})) \leq L_F d(x, \bar{x})$, and $z \in G(x)$ with $d(z, G(\bar{x})) \leq L_G d(x, \bar{x})$, such that $w = y + z$. Then $F + G$ is $(L_F + L_G)$ -calm at $(\bar{x}, \bar{y} + \bar{z})$.*

Remark that if F is L_F -calm at (\bar{x}, \bar{y}) , G is L_G -calm at (\bar{x}, \bar{z}) , and (F, G) is locally sum-stable around $(\bar{x}, \bar{y}, \bar{z})$, we are in the setting of the previous proposition. Other situations are presented in the following example.

Example 3.8 1. Consider the multifunctions $F, G : \mathbb{R} \rightrightarrows \mathbb{R}$, given by

$$F(x) := \begin{cases} [0, 2], & \text{if } x \in \mathbb{R} \setminus \{0\} \\ [0, 1], & \text{if } x = 0 \end{cases}$$

and

$$G(x) := \begin{cases} [0, 2], & \text{if } x \in \mathbb{R} \setminus \{0\} \\ [1, 2], & \text{if } x = 0, \end{cases}$$

which are not calm at $(0, 1)$. Then the multifunction $F + G : \mathbb{R} \rightrightarrows \mathbb{R}$, given by

$$(F + G)(x) = \begin{cases} [0, 4], & \text{if } x \in \mathbb{R} \setminus \{0\} \\ [1, 3], & \text{if } x = 0, \end{cases}$$

is calm at $(0, 2)$.

2. Consider the multifunctions $F, G : [-1, 1] \rightrightarrows \mathbb{R}$, given by

$$F(x) := \begin{cases} [0, x + 1], & \text{if } x \in [-1, 0) \\ \{0\} \cup \left[\frac{1}{2}, 1\right], & \text{if } x = 0 \\ [0, 1 - x], & \text{if } x \in (0, 1], \end{cases}$$

which is not calm at $(0, 0)$, and

$$G(x) := \begin{cases} \{-1 - x, 0\}, & \text{if } x \in [-1, 0) \\ [-1, 0], & \text{if } x = 0 \\ \{-1 + x, 0\}, & \text{if } x \in (0, 1], \end{cases}$$

which is calm at $(0, 0)$. Then the multifunction $F + G : \mathbb{R} \rightrightarrows \mathbb{R}$, given by

$$(F + G)(x) = \begin{cases} [-x - 1, x + 1], & \text{if } x \in [-1, 0) \\ [-1, 1], & \text{if } x = 0 \\ [x - 1, 1 - x], & \text{if } x \in (0, 1] \end{cases}$$

is calm at $(0, 0)$. Remark that (F, G) is not locally sum-stable around $(0, 0, 0)$.

3.3 Applications to variational systems

This subsection plays a leading role in this work, since here we put together all the facts collected by now and we use them in order to study variational systems.

To begin, we adapt the definition of local-sum stability to the parametric case.

Definition 3.9 ([19, Definition 4.9]) *Let $F : X \times P \rightrightarrows Y$, $G : X \rightrightarrows Y$ be two multifunctions and $(\bar{x}, \bar{p}, \bar{y}, \bar{z}) \in X \times P \times Y \times Y$ such that $\bar{y} \in F(\bar{x}, \bar{p})$, $\bar{z} \in G(\bar{x})$. We say that the multifunction (F, G) is locally sum-stable around $((\bar{x}, \bar{p}), \bar{y}, \bar{z})$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $(x, p) \in B(\bar{x}, \delta) \times B(\bar{p}, \delta)$ and every $w \in (F_p + G)(x) \cap B(\bar{y} + \bar{z}, \delta)$, there exist $y \in F_p(x) \cap B(\bar{y}, \varepsilon)$ and $z \in G(x) \cap B(\bar{z}, \varepsilon)$ such that $w = y + z$.*

Also, Lemma 3.6 has the following variant in the parametric case.

Lemma 3.10 *Let $F : X \times P \rightrightarrows Y$, $G : X \rightrightarrows Y$ be two multifunctions. Suppose that F is calm with respect to x uniformly in p at $((\bar{x}, \bar{p}), \bar{y}) \in \text{Gr } F$, that G is calm at $(\bar{x}, \bar{z}) \in \text{Gr } G$ and that (F, G) is locally sum-stable around $((\bar{x}, \bar{p}), \bar{y}, \bar{z})$. Then the multifunction $H : X \times P \rightrightarrows Y$ given by $H(x, p) := F(x, p) + G(x)$ is calm with respect to x uniformly in p at $((\bar{x}, \bar{p}), \bar{y} + \bar{z})$. Moreover, the following relation holds true*

$$\widehat{\text{clm}}_x H((\bar{x}, \bar{p}), \bar{y} + \bar{z}) \leq \widehat{\text{clm}}_x F((\bar{x}, \bar{p}), \bar{y}) + \text{clm } G(\bar{x}, \bar{z}). \quad (3.9)$$

Proof. Adapt the line of the proof of Lemma 3.6. \square

In the next theorems we get at-point regularity results for S defined by (3.1), where H takes the form

$$H(x, p) := F(x, p) + G(x). \quad (3.10)$$

The case of around-point regularity was considered in [1], [19], while particular situations for at-point regularity are studied in [1], [2].

Theorem 3.11 *Let X, Y, P be Banach spaces, $F : X \times P \rightrightarrows Y$, $G : X \rightrightarrows Y$ be two set-valued maps and $(\bar{x}, \bar{p}, \bar{y}) \in X \times P \times Y$ such that $\bar{y} \in F(\bar{x}, \bar{p})$ and $-\bar{y} \in G(\bar{x})$. Suppose that the following assumptions are satisfied:*

- (i) (F, G) is locally sum-stable around $((\bar{x}, \bar{p}), \bar{y}, -\bar{y})$;

(ii) F is calm with respect to x uniformly in p at $((\bar{x}, \bar{p}), \bar{y})$;

(iii) $F_{\bar{x}}$ is metrically regular around (\bar{p}, \bar{y}) ;

(iv) G is calm at $(\bar{x}, -\bar{y})$.

Then S is metrically subregular at (\bar{p}, \bar{x}) . Moreover, the next relation holds

$$\text{subreg } S(\bar{p}, \bar{x}) \leq \text{reg } F_{\bar{x}}(\bar{p}, \bar{y}) \cdot [\widehat{\text{clm}}_x F((\bar{x}, \bar{p}), \bar{y}) + \text{clm } G(\bar{x}, -\bar{y})]. \quad (3.11)$$

Proof. Using Lemma 3.10, we know that H given by (3.10) is calm with respect to x uniformly in p at $((\bar{x}, \bar{p}), 0)$ and the relation (3.9) holds for $\bar{z} := -\bar{y}$.

Using (iii), which is equivalent to the linear openness of $F_{\bar{x}}$ around (\bar{p}, \bar{y}) , one can find $\alpha, L > 0$ such that, for every $(p, y) \in \text{Gr } F_{\bar{x}} \cap [B(\bar{p}, \alpha) \times B(\bar{y}, \alpha)]$ and every $\rho \in (0, \alpha)$,

$$B(y, L\rho) \subset F_{\bar{x}}(B(p, \rho)).$$

Also, from (i), for α found before, there is $\delta \in (0, \alpha)$ such that, for every $(x, p) \in B(\bar{x}, \delta) \times B(\bar{p}, \delta)$ and every $w \in H(x, p) \cap B(0, \delta)$, one can find $y \in F(x, p) \cap B(\bar{y}, \alpha)$ and $z \in G(x) \cap B(-\bar{y}, \alpha)$ such that $w = y + z$.

Fix now $\tau < \min\{\alpha, L^{-1}\delta\}$, and take $\rho \in (0, \tau)$, $p \in B(\bar{p}, \delta) \cap H_{\bar{x}}^{-1}(B(0, L\rho))$. Then there exists $w \in B(0, L\rho)$ such that $w \in H(\bar{x}, p)$. Then $w \in B(0, \delta)$, so using the local sum-stability of (F, G) , one can find $y \in F(\bar{x}, p) \cap B(\bar{y}, \alpha)$ and $z \in G(\bar{x}) \cap B(-\bar{y}, \alpha)$ such that $w = y + z$. Then

$$0 \in B(w, L\rho) = B(y, L\rho) + z \subset F_{\bar{x}}(B(p, \rho)) + z \subset H_{\bar{x}}(B(p, \rho)).$$

As consequence, $H_{\bar{x}}$ is linearly pseudo-open at $(\bar{p}, 0)$, with modulus $(\text{reg } F_{\bar{x}}(\bar{p}, \bar{y}))^{-1}$. The conclusion now follows from the second part of Theorem 3.1. \square

A natural question which arises when one looks at Theorem 3.11 is if the assumption (iii) cannot be weakened, supposing for example just the metric subregularity of $F_{\bar{x}}$ at (\bar{p}, \bar{y}) . The next example clarifies this aspect, showing that if one replaces the metric regularity of $F_{\bar{x}}$ with its metric subregularity, the conclusion of Theorem 3.11 is not satisfied in general.

Example 3.12 Consider the multifunctions $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ and $G : \mathbb{R} \rightrightarrows \mathbb{R}^2$ given by

$$F(x, p) := \{(0, p)\} \cup \left\{ \left(\frac{1}{n^2}, p - \frac{1}{n^2} \right) \mid n \in \mathbb{N} \setminus \{0\} \right\} \text{ and}$$

$$G(x) := \{(x, 0)\} \cup \left\{ \left(x + \frac{1}{n^3}, \frac{1}{n} \right) \mid n \in \mathbb{N} \setminus \{0\} \right\}.$$

Also, fix $\bar{x} := 0$, $\bar{p} := 0$, $\bar{y} := (0, 0)$. Then $H : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ is given by

$$H(x, p) = \{(x, p)\} \cup \left\{ \left(x + \frac{1}{n^2}, p - \frac{1}{n^2} \right) \mid n \in \mathbb{N} \setminus \{0\} \right\} \cup$$

$$\left\{ \left(x + \frac{1}{n^3}, p + \frac{1}{n} \right) \mid n \in \mathbb{N} \setminus \{0\} \right\} \cup$$

$$\left\{ \left(x + \frac{1}{n^3} + \frac{1}{m^2}, p - \frac{1}{m^2} + \frac{1}{n} \right) \mid n, m \in \mathbb{N} \setminus \{0\} \right\}.$$

Then,

$$S^{-1}(x) = \begin{cases} 0, & \text{if } x = 0 \\ \frac{1}{n^2}, & \text{if } x = -\frac{1}{n^2}, n \in \mathbb{N} \setminus \{0\} \\ -\frac{1}{n}, & \text{if } x = -\frac{1}{n^3}, n \in \mathbb{N} \setminus \{0\} \\ \frac{1}{m^2} - \frac{1}{n}, & \text{if } x = -\frac{1}{n^3} - \frac{1}{m^2}, m, n \in \mathbb{N} \setminus \{0\} \\ \emptyset, & \text{otherwise.} \end{cases}$$

Let us remark that $S^{-1}(0) = \{0\}$. Also, one can prove that S^{-1} is not calm at $(0, 0)$. If S^{-1} would be calm at $(0, 0)$, it should exist $l, \alpha > 0$ such that, for every $x \in B(0, \alpha)$,

$$S^{-1}(x) \cap B(0, \alpha) \subset S^{-1}(0) + l|x|[-1, 1].$$

Take $x = -\frac{1}{n^3} - \frac{1}{m^2}$, with $m, n \in \mathbb{N} \setminus \{0\}$, such that $\frac{1}{n^3} + \frac{1}{m^2} < \alpha$ and $\left| \frac{1}{m^2} - \frac{1}{n} \right| < \alpha$. One should have

$$\left| \frac{1}{m^2} - \frac{1}{n} \right| < l \left(\frac{1}{n^3} + \frac{1}{m^2} \right),$$

for every m, n sufficiently large. But for $m = n$, one should have that

$$\begin{aligned} \frac{1}{n} - \frac{1}{n^2} &< l \left(\frac{1}{n^3} + \frac{1}{n^2} \right), \text{ or} \\ \frac{(n-1)n}{n+1} &< l \end{aligned}$$

for every n sufficiently large, which is absurd. In conclusion, S^{-1} is not calm at $(0, 0)$, or S is not metrically subregular at $(0, 0)$.

Let us prove now that F_0 is metrically subregular at $(0, (0, 0))$, but F is not metrically regular around $(0, (0, 0))$.

As one can see,

$$F_0^{-1}(u, v) = \begin{cases} p, & \text{if } (u, v) = (0, p) \text{ or } (u, v) = \left(\frac{1}{n^2}, p - \frac{1}{n^2} \right), n \in \mathbb{N} \setminus \{0\} \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then $F_0^{-1}(0, 0) = \{0\}$. Also, let's prove that there exists $\beta > 0$ such that, for every $(u, v) \in B((0, 0), \beta)$,

$$F_0^{-1}(u, v) \cap B(0, \beta) \subset F_0^{-1}(0, 0) + \|(u, v)\|[-1, 1]. \quad (3.12)$$

Indeed, take arbitrary $\beta > 0$ and $(u, v) \in B((0, 0), \beta)$ such that $F_0^{-1}(u, v) \neq \emptyset$. Then $(u, v) = (0, p)$, in which case relation (3.12) reduces to

$$|p| \leq \|(0, p)\|,$$

or $(u, v) = \left(\frac{1}{n^2}, p - \frac{1}{n^2} \right)$, in which case relation (3.12) becomes

$$|p| \leq \frac{1}{n^2} + \left| p - \frac{1}{n^2} \right|,$$

which is obviously true from the triangle inequality.

But, as one can easily see, $F_0^{-1}(u', v')$ can be empty for (u', v') arbitrarily close to $(0, 0)$, hence the relation

$$F_0^{-1}(u, v) \cap B(0, \beta) \subset F_0^{-1}(u', v') + m \|(u, v) - (u', v')\| [-1, 1]$$

cannot be true for every (u', v') in a neighborhood of $(0, 0)$. As consequence, F_0^{-1} does not have the Aubin property $((0, 0), 0)$, or F_0 is not metrically regular around $(0, (0, 0))$.

Being constant with respect to x , F is obviously calm with respect to x , uniformly in p at $((0, 0), (0, 0))$ with modulus 1. Also, one can easily see that G is calm at $(0, (0, 0))$ with modulus 1.

Let us now prove that (F, G) is locally sum-stable around $((0, 0), (0, 0), (0, 0))$. Indeed, take arbitrary $\varepsilon > 0$. Pick $\delta < \min \left\{ \frac{2}{7}\varepsilon, \frac{18}{343}\varepsilon^3 \right\}$, and take $x \in \left(-\frac{\delta}{2}, \frac{\delta}{2} \right)$, $p \in \left(-\frac{\delta}{2}, \frac{\delta}{2} \right)$ and $w \in H(x, p) \cap B((0, 0), \delta)$. We have four possibilities:

1. If $w = (x, p)$, then the conclusion easily follows.

2. If $w = \left(x + \frac{1}{n^2}, p - \frac{1}{n^2} \right)$, with $\left| x + \frac{1}{n^2} \right| + \left| p - \frac{1}{n^2} \right| < \delta$, then $\frac{1}{n^2} < \frac{3}{2}\delta$, because otherwise

$$\left| x + \frac{1}{n^2} \right| \geq \frac{1}{n^2} - |x| \geq \frac{3}{2}\delta - \frac{1}{2}\delta = \delta,$$

which is absurd. Then w can be written as $\left(\frac{1}{n^2}, p - \frac{1}{n^2} \right) + (x, 0)$, with $\left(\frac{1}{n^2}, p - \frac{1}{n^2} \right) \in F(x, p)$, $(x, 0) \in G(x)$, and

$$\begin{aligned} \left\| \left(\frac{1}{n^2}, p - \frac{1}{n^2} \right) \right\| &= \frac{1}{n^2} + \left| p - \frac{1}{n^2} \right| < \frac{5}{2}\delta < \varepsilon, \\ \|(x, 0)\| &= |x| < \frac{1}{2}\delta < \varepsilon. \end{aligned}$$

3. If $w = \left(x + \frac{1}{n^3}, p + \frac{1}{n} \right)$, with $\left| x + \frac{1}{n^3} \right| + \left| p + \frac{1}{n} \right| < \delta$, then $\frac{1}{n} < \frac{3}{2}\delta$, because, again, in the opposite situation one should have

$$\left| p + \frac{1}{n} \right| \geq \frac{1}{n} - |p| \geq \frac{3}{2}\delta - \frac{1}{2}\delta = \delta,$$

which is absurd. Then w can be written as $(0, p) + \left(x + \frac{1}{n^3}, \frac{1}{n} \right)$, with $(0, p) \in F(x, p)$, $\left(x + \frac{1}{n^3}, \frac{1}{n} \right) \in G(x)$, and

$$\begin{aligned} \|(0, p)\| &= |p| < \frac{1}{2}\delta < \varepsilon, \\ \left\| \left(x + \frac{1}{n^3}, \frac{1}{n} \right) \right\| &= \left| x + \frac{1}{n^3} \right| + \frac{1}{n} < \frac{5}{2}\delta < \varepsilon. \end{aligned}$$

4. Finally, if $w = \left(x + \frac{1}{n^3} + \frac{1}{m^2}, p - \frac{1}{m^2} + \frac{1}{n} \right)$, with $\left| x + \frac{1}{n^3} + \frac{1}{m^2} \right| + \left| p - \frac{1}{m^2} + \frac{1}{n} \right| < \delta$, then, as above, one can prove that $\left| \frac{1}{n^3} + \frac{1}{m^2} \right| < \frac{3}{2}\delta$. But this means that $\frac{1}{n} < \sqrt[3]{\frac{3}{2}\delta}$ and $\frac{1}{m} <$

$\sqrt{\frac{3}{2}}\delta$. Then w can be written as $\left(\frac{1}{m^2}, p - \frac{1}{m^2}\right) + \left(x + \frac{1}{n^3}, \frac{1}{n}\right)$, with $\left(\frac{1}{m^2}, p - \frac{1}{m^2}\right) \in F(x, p)$, $\left(x + \frac{1}{n^3}, \frac{1}{n}\right) \in G(x)$, and

$$\begin{aligned} \left\| \left(\frac{1}{m^2}, p - \frac{1}{m^2} \right) \right\| &= \frac{1}{m^2} + \left| p - \frac{1}{m^2} \right| \leq \frac{2}{m^2} + |p| < 3\delta + \frac{\delta}{2} = \frac{7}{2}\delta < \varepsilon, \\ \left\| \left(x + \frac{1}{n^3}, \frac{1}{n} \right) \right\| &= \left| x + \frac{1}{n^3} \right| + \frac{1}{n} \leq |x| + \frac{1}{n^3} + \frac{1}{n} < \frac{\delta}{2} + \frac{3}{2}\delta + \sqrt[3]{\frac{3}{2}}\delta < \varepsilon. \end{aligned}$$

In conclusion, (F, G) is locally sum-stable around $((0, 0), (0, 0), (0, 0))$.

Let us finally prove that, for every $L > 0$, there exist ρ arbitrary small, p arbitrary small, and $w \in H_0(p) \cap D((0, 0), \rho)$ such that w cannot be written as $y + z$, with $y \in F_0(p) \cap D((0, 0), L\rho)$ and $z \in G(0)$. Indeed, take $\rho = \frac{1}{n^3} + \frac{2}{n^2}$, $p = -\frac{1}{n}$ arbitrary small, and $w = \left(\frac{1}{n^3} + \frac{1}{n^2}, -\frac{1}{n^2}\right) \in H_0(p)$ such that $\|w\| = \rho$. This w can only be obtained as $\left(\frac{1}{n^2}, -\frac{1}{n} - \frac{1}{n^2}\right) + \left(\frac{1}{n^3}, \frac{1}{n}\right)$, with $\left(\frac{1}{n^2}, -\frac{1}{n} - \frac{1}{n^2}\right) \in F_0(p)$, and $\left(\frac{1}{n^3}, \frac{1}{n}\right) \in G(0)$. Let us prove this assertion. Suppose there exist $k, m \in \mathbb{N} \setminus \{0\}$ such that

$$\begin{cases} \frac{1}{k^3} + \frac{1}{m^2} = \frac{1}{n^3} + \frac{1}{n^2} \\ -\frac{1}{m^2} + \frac{1}{k} - \frac{1}{n} = -\frac{1}{n^2} \end{cases}.$$

This means that $\frac{1}{k^3} + \frac{1}{k} = \frac{1}{n^3} + \frac{1}{n}$. As the function $k \mapsto \frac{1}{k^3} + \frac{1}{k}$ is strictly decreasing, the only solution of this equation is $k = n$. But this shows also that $m = n$.

Now, suppose that there is $L > 0$ such that $\left\| \left(\frac{1}{n^2}, -\frac{1}{n} - \frac{1}{n^2} \right) \right\| < L\rho = L \left(\frac{1}{n^3} + \frac{2}{n^2} \right)$, for every n sufficiently large. This means that

$$\begin{aligned} \frac{2}{n^2} + \frac{1}{n} &< L \left(\frac{1}{n^3} + \frac{2}{n^2} \right), \text{ or} \\ \frac{(n+2)n}{2n+1} &< L, \end{aligned}$$

for every n sufficiently large, which is absurd. \square

Notice that the phenomenon described at the final of the previous example (which in fact generated it) seems to be the reason for which only the subregularity condition for $F_{\overline{x}}$ is not sufficient in getting the conclusion of Theorem 3.11. In fact, the problem was the impossibility to obtain a (linear) correspondence needed to link the closeness between w and \overline{w} with the one between y and \overline{y} . A possible solution could be to introduce a notion of partial linear sum-stability (in which to ask for this linear correspondence, but to maintain only y close to \overline{y}). We preferred to avoid this approach for clarity. The next example, which seems to be more simple than the one given before (at least in verifying the conditions for the involved objects) mainly addresses the same questions, but the phenomenon described before is less visible. For this reason, we keep them both.

Example 3.13 Consider the multifunctions $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}$ and $G : \mathbb{R} \rightrightarrows \mathbb{R}$ given by

$$F(x, p) := \begin{cases} [|p|, +\infty) \cap \mathbb{Q}, & \text{if } x = 0 \\ [|p|, +\infty), & \text{if } x \neq 0, \end{cases}$$

$$G(x) := \begin{cases} [-1, 0] \setminus \mathbb{Q} \cup \{0\}, & \text{if } x = 0 \\ [-1, 0], & \text{if } x \neq 0 \end{cases}$$

and take $\bar{x} = \bar{p} = \bar{y} = \bar{z} := 0$. Then F and G here share all the properties from the previous example and, again, S is not metrically subregular at $(0, 0)$.

The next theorem, previously given in [19], is a sort of Lyusternik-Graves type result, and has the role to precisely specify the constants involved in the openness property. This will be important in the development of other subsequent results.

Theorem 3.14 ([19, Theorem 3.3]) *Let X, Y be Banach spaces, $F_1 : X \rightrightarrows Y$ and $F_2 : Y \rightrightarrows X$ be two multifunctions and $(\bar{x}, \bar{y}, \bar{z}) \in X \times Y \times Y$ such that $(\bar{x}, \bar{y}) \in \text{Gr } F_1$ and $(\bar{z}, \bar{x}) \in \text{Gr } F_2$. Suppose that the following assumptions are satisfied:*

- (i) *$\text{Gr } F_1$ is locally closed around (\bar{x}, \bar{y}) , so there exist $\alpha_1, \beta_1 > 0$ such that $\text{Gr } F_1 \cap [D(\bar{x}, \alpha_1) \times D(\bar{y}, \beta_1)]$ is closed;*
- (ii) *$\text{Gr } F_2$ is locally closed around (\bar{z}, \bar{x}) , so there exist $\alpha_2, \beta_2 > 0$ such that $\text{Gr } F_2 \cap [D(\bar{z}, \beta_2) \times D(\bar{x}, \alpha_2)]$ is closed;*
- (iii) *there exist $L, r_1, s_1 > 0$ such that, for every $(x', y') \in \text{Gr } F_1 \cap [B(\bar{x}, r_1) \times B(\bar{y}, s_1)]$, F_1 is L -open at (x', y') ;*
- (iv) *there exist $M, r_2, s_2 > 0$ such that, for every $(v', u') \in \text{Gr } F_2 \cap [B(\bar{z}, s_2) \times B(\bar{x}, r_2)]$, F_2 is M -open at (v', u') ;*
- (v) *$LM > 1$.*

Then for every $\rho \in (0, \varepsilon)$, where $\varepsilon := \min\{\alpha_1, \alpha_2, L^{-1}\beta_1, M\beta_2, r_1, r_2, L^{-1}s_1, Ms_2\}$,

$$B(\bar{y} - \bar{z}, (L - M^{-1})\rho) \subset (F_1 - F_2^{-1})(B(\bar{x}, \rho)).$$

We are able to present now a fixed-point assertion, given in the parametric form, which follows the path opened by Arutyunov in a series of papers ([3]–[6]), and after that continued by Donchev and Frankowska ([13], [14]), Ioffe ([24]), and Durea and Strugariu ([20], [21]), where links between fixed-point theorems and Lyusternik-Graves type results are provided. The inequality from the conclusion of Theorem 3.15 could be formulated in some different ways, as is done in [13, Theorem 7], [20, Theorem 4.4]. We prefer to obtain it just in the form we need in the sequel. For this, consider the multifunctions $\Phi : X \times P \rightrightarrows Y$, $\Psi : X \rightrightarrows Y$, and take the implicit multifunction $S : P \rightrightarrows X$ as

$$S(p) := (\Phi(\cdot, p) - \Psi)^{-1}(0) = \{x \in X \mid 0 \in \Phi(x, p) - \Psi(x)\}$$

$$= \{x \in X \mid \Phi(x, p) \cap \Psi(x) \neq \emptyset\} = \text{Fix}(\Phi(\cdot, p)^{-1}\Psi).$$

Although the proof of the next result has some common points to the one of the first part of Theorem 3.1 (i), it involves some more technicalities. For this reason, we present it in full extent.

Theorem 3.15 *Let X, Y be Banach spaces, P be a metric space, $\Phi : X \times P \rightrightarrows Y$ and $\Psi : X \rightrightarrows Y$ be multifunctions and $(\bar{x}, \bar{p}, \bar{y}) \in X \times P \times Y$ such that $((\bar{x}, \bar{p}), \bar{y}) \in \text{Gr } \Phi$ and $(\bar{x}, \bar{y}) \in \text{Gr } \Psi$. Suppose that the following assumptions are satisfied:*

- (i) $\text{Gr } \Phi_p$ is locally closed around (\bar{x}, \bar{y}) uniformly for p in a neighborhood of \bar{p} ;
- (ii) $\text{Gr } \Psi$ is locally closed around (\bar{x}, \bar{y}) ;
- (iii) $(\Phi, -\Psi)$ is locally sum-stable around $((\bar{x}, \bar{p}), \bar{y}, -\bar{y})$;
- (iv) Φ has the Aubin property with respect to x uniformly in p around $((\bar{x}, \bar{p}), \bar{y})$ with constant $l > 0$;

- (v) Ψ is metrically regular with constant $m > 0$ around (\bar{x}, \bar{y}) ;
- (vi) $lm < 1$.

Then there exist $\alpha, \beta > 0$ such that for any $(x, p) \in B(\bar{x}, \alpha) \times B(\bar{p}, \alpha)$ one has that

$$d(x, S(p)) \leq (m^{-1} - l)^{-1} d(0, [\Phi(x, p) - \Psi(x)] \cap B(0, \beta)). \quad (3.13)$$

Proof. Using (i) and (ii), one can pick $\gamma > 0$ such that, for every $p \in B(\bar{p}, \gamma)$, $\text{Gr } \Phi_p \cap [D(\bar{x}, \gamma) \times D(\bar{y}, \gamma)]$ is closed and $\text{Gr } \Psi \cap [D(\bar{x}, \gamma) \times D(\bar{y}, \gamma)]$ is closed.

Also, by (iv), there exist $r \in (0, \gamma)$ such that, for every $p \in B(\bar{p}, r)$, and every $x, x' \in B(\bar{x}, r)$,

$$e(\Phi_p(x) \cap B(\bar{y}, r), \Phi_p(x')) \leq ld(x, x'). \quad (3.14)$$

Take now $x \in B(\bar{x}, 2^{-1}r)$ and $y \in \Phi_p(x) \cap B(\bar{y}, r)$. Then for every $x' \in B(x, 2^{-1}r)$, we get by (3.14) that

$$d(y, \Phi_p(x')) \leq ld(x, x'),$$

which proves that for every $p \in B(\bar{p}, r)$, Φ_p is l -pseudocalm at every $(x, y) \in \text{Gr } \Phi_p \cap [B(\bar{x}, 2^{-1}r) \times B(\bar{y}, r)]$. In virtue of Proposition 2.5, we deduce that for every $p \in B(\bar{p}, r)$, Φ_p^{-1} is l^{-1} -open at every $(y, x) \in \text{Gr } \Phi_p^{-1} \cap [B(\bar{y}, r) \times B(\bar{x}, 2^{-1}r)]$.

From (v), we know that there exist $t > 0$ such that, for every $(u, v) \in \text{Gr } \Psi \cap [B(\bar{x}, t) \times B(\bar{y}, t)]$, Ψ is metrically hemiregular at (u, v) with constant m , hence is open at linear rate m^{-1} at (u, v) .

From the local sum-stability of $(\Phi, -\Psi)$ around $((\bar{x}, \bar{p}), \bar{y}, -\bar{y})$, using $\min\{2^{-1}r, 2^{-1}t\}$ instead of ε , one can find $\delta > 0$ such that, for every $(x, p) \in B(\bar{x}, \delta) \times B(\bar{p}, \delta)$ and every $w \in (\Phi_p - \Psi)(x) \cap B(0, \delta)$, there exist $y \in \Phi_p(x) \cap B(\bar{y}, \varepsilon)$ and $z \in \Psi(x) \cap B(\bar{y}, \varepsilon)$ such that $w = y - z$.

Take now $\rho \in (0, \min\{(m^{-1} - l)^{-1}\delta, 2^{-1}\gamma, 2^{-1}m\gamma, 2^{-1}l^{-1}\gamma, 2^{-1}t, 4^{-1}r, 2^{-1}mt, 2^{-1}l^{-1}r\})$ and define $\beta := (m^{-1} - l)\rho$. Also, denote by $\alpha := \min\{2^{-1}\gamma, 4^{-1}r\}$, and take $(x, p) \in B(\bar{x}, \alpha) \times B(\bar{p}, \alpha)$.

If $[\Phi(x, p) - \Psi(x)] \cap B(0, \beta) = \emptyset$ or $0 \in [\Phi(x, p) - \Psi(x)] \cap B(0, \beta)$, then (3.13) trivially holds. Suppose next that $0 \notin [\Phi(x, p) - \Psi(x)] \cap B(0, \beta)$. Then, for every $\xi > 0$, one can find $w_\xi \in [\Phi(x, p) - \Psi(x)] \cap B(0, \gamma)$ such that

$$0 < \|w_\xi\| < d(0, [\Phi(x, p) - \Psi(x)] \cap B(0, \beta)) + \xi. \quad (3.15)$$

Obviously, $d(0, [\Phi(x, p) - \Psi(x)] \cap B(0, \beta)) < (m^{-1} - l)\rho$, hence for $\xi > 0$ sufficiently small, $d(0, [\Phi(x, p) - \Psi(x)] \cap B(0, \beta)) + \xi < (m^{-1} - l)\rho$. As consequence, we get from (3.15) that

$$0 \in B(w_\xi, d(0, [\Phi(x, p) - \Psi(x)] \cap B(0, \beta)) + \xi) \subset B(w_\xi, (m^{-1} - l)\rho) = B(w_\xi, \beta). \quad (3.16)$$

Applying the local sum-stability, one can find $y_\xi \in \Phi(x, p) \cap B(\bar{y}, 2^{-1}r)$ and $z_\xi \in \Psi(x) \cap B(\bar{y}, 2^{-1}t)$ such that $w_\xi = y_\xi - z_\xi$. Observe also that $B(y_\xi, 2^{-1}r) \subset B(\bar{y}, r)$ and $B(z_\xi, 2^{-1}t) \subset B(\bar{y}, t)$.

Denote now $\alpha_1 := \beta_1 := \alpha_2 := \beta_2 := 2^{-1}\gamma$, $r_1 := 2^{-1}t$, $s_1 := 2^{-1}t$, $r_2 := 4^{-1}r$, $s_2 := 2^{-1}r$. Summarizing,

$$\text{Gr } \Psi \cap [D(x, \alpha_1) \times D(z_\xi, \beta_1)] \subset \text{Gr } \Psi \cap [D(\bar{x}, \gamma) \times D(\bar{y}, \gamma)] \text{ is closed,}$$

$$\text{Gr } \Phi_p^{-1} \cap [D(y_\xi, \beta_2) \times D(x, \alpha_2)] \subset \text{Gr } \Phi_p^{-1} \cap [D(\bar{y}, \gamma) \times D(\bar{x}, \gamma)] \text{ is closed,}$$

$$\Psi \text{ is } m^{-1} \text{ - open at every } (u', v') \in \text{Gr } \Psi \cap [B(x, r_1) \times B(z_\xi, s_1)] \subset \text{Gr } \Psi \cap [B(\bar{x}, t) \times B(\bar{y}, t)],$$

$$\Phi_p^{-1} \text{ is } l^{-1} \text{ - open at every } (y', x') \in \text{Gr } \Phi_p^{-1} \cap [B(y_\xi, s_2) \times B(x, r_2)] \subset \text{Gr } \Phi_p^{-1} \cap [B(\bar{y}, r) \times B(\bar{x}, 2^{-1}r)],$$

$$l^{-1}m^{-1} > 1.$$

We can apply now Theorem 3.14 for $\Psi, \Phi_p^{-1}, (x, z_\xi) \in \text{Gr } \Psi, (y_\xi, x) \in \text{Gr } \Phi_p^{-1}$ and

$$\rho_0 := (m^{-1} - l)^{-1}(d(0, [\Phi(x, p) - \Psi(x)] \cap B(0, \beta)) + \xi) < \rho < \min\{\alpha_1, \alpha_2, m\beta_1, l^{-1}\beta_2, r_1, r_2, ms_1, l^{-1}s_2\}$$

to obtain that

$$0 \in B(z_\xi - y_\xi, d(0, [\Phi(x, p) - \Psi(x)] \cap B(0, \beta)) + \xi) \subset (\Psi - \Phi_p)(B(x, \rho_0)).$$

Using (3.16), we obtain that $0 \in (\Psi - \Phi_p)(B(x, \rho_0))$, so there exists $\tilde{x} \in B(x, \rho_0)$ such that $0 \in \Psi(\tilde{x}) - \Phi(\tilde{x}, p)$ or, equivalently, $\tilde{x} \in S(p)$. Hence

$$d(x, S(p)) \leq d(x, \tilde{x}) < \rho_0 = (m^{-1} - l)^{-1}(d(0, [\Phi(x, p) - \Psi(x)] \cap B(0, \beta)) + \xi).$$

Making $\xi \rightarrow 0$, one gets (3.13). \square

Notice that in Theorem 3.15 the conclusion reads as a Robinson regularity of S (see [30], [31]). Remark also that in order to get (3.13) the property of local sum-stability naturally arises, in contrast to other inequalities involving the Robinson regularity of S (compare to [13, Theorem 7], [20, Theorem 4.4]).

Finally, Theorem 3.15 gives us the possibility to formulate a result concerning the calmness of the implicit multifunction S associated to a parametric variational system.

Theorem 3.16 *Let X, Y, P be Banach spaces, $F : X \times P \rightrightarrows Y, G : X \rightrightarrows Y$ be two set-valued maps and $(\bar{x}, \bar{p}, \bar{y}) \in X \times P \times Y$ such that $\bar{y} \in F(\bar{x}, \bar{p})$ and $-\bar{y} \in G(\bar{x})$. Suppose that the following assumptions are satisfied:*

- (i) (F, G) is locally sum-stable with respect to x uniformly in p around $((\bar{x}, \bar{p}), \bar{y}, -\bar{y})$;
- (ii) $\text{Gr } F_p$ is locally closed around (\bar{x}, \bar{y}) uniformly for p in a neighborhood of \bar{p} ;
- (iii) $\text{Gr } G$ is locally closed around $(\bar{x}, -\bar{y})$;
- (iv) F has the Aubin property with respect to x , uniformly in p , around $((\bar{x}, \bar{p}), \bar{y})$;
- (v) F is calm with respect to p , uniformly in x , at $((\bar{x}, \bar{p}), \bar{y})$;
- (vi) G is metrically regular around $(\bar{x}, -\bar{y})$;
- (vii) $\widehat{\text{lip}}_x F((\bar{x}, \bar{p}), \bar{y}) \cdot \text{reg } G(\bar{x}, -\bar{y}) < 1$.

Then S is calm at (\bar{p}, \bar{x}) . Moreover, the next relation is satisfied

$$\text{clm } S(\bar{p}, \bar{x}) \leq \frac{\text{reg } G(\bar{x}, -\bar{y}) \cdot \widehat{\text{clm}}_p F((\bar{x}, \bar{p}), \bar{y})}{1 - \widehat{\text{lip}}_x F((\bar{x}, \bar{p}), \bar{y}) \cdot \text{reg } G(\bar{x}, -\bar{y})}. \quad (3.17)$$

Proof. Take $m > \text{reg } G(\bar{x}, -\bar{y})$ and $l > \widehat{\text{lip}}_x F((\bar{x}, \bar{p}), \bar{y})$ such that $m \cdot l < 1$. Apply now Theorem 3.15 for F and G instead of Φ and $-\Psi$, respectively, to get that there exist $\alpha, \beta > 0$ such that for any $(x, p) \in B(\bar{x}, \alpha) \times B(\bar{p}, \alpha)$,

$$d(x, S(p)) \leq (m^{-1} - l)^{-1} d(0, [F(x, p) + G(x)] \cap B(0, \beta)). \quad (3.18)$$

Next, use (v) to get that there exist $\gamma, k > 0$ such that for every $(x, p) \in B(\bar{x}, \gamma) \times B(\bar{p}, \gamma)$,

$$e(F_x(p) \cap B(\bar{y}, \gamma), F_x(\bar{p})) \leq kd(p, \bar{p}). \quad (3.19)$$

From the local sum-stability of (F, G) for γ instead of ε , there is $\delta \in (0, \gamma)$ such that the assertion from Definition 3.9 is true. Take now arbitrary $x \in B(\bar{x}, \delta), p \in B(\bar{p}, \delta)$ and $w \in H(x, p) \cap B(0, \delta)$,

where H is given by relation (3.10). Then there exist $y \in F_x(p) \cap B(\bar{y}, \gamma)$ and $z \in G(x) \cap B(-\bar{y}, \gamma)$ such that $w = y + z$. By (3.19), we obtain

$$\begin{aligned} d(w, H(x, \bar{p})) &= d(y + z, H(x, \bar{p})) \leq d(y + z, F(x, \bar{p}) + z) \\ &= d(y, F(x, \bar{p})) \leq kd(p, \bar{p}). \end{aligned}$$

As w was arbitrarily taken from $H(x, p) \cap B(0, \delta)$, it follows that H is calm with respect to p uniformly in x at $((\bar{x}, \bar{p}), 0)$. As k can be chosen arbitrarily close to $\widehat{\text{clm}}_p F((\bar{x}, \bar{p}), \bar{y})$, we deduce that $\widehat{\text{clm}}_p F((\bar{x}, \bar{p}), \bar{y}) \geq \widehat{\text{clm}}_p H((\bar{x}, \bar{p}), 0)$.

In conclusion, the relation of the type (3.2) follows from (3.18), and H is calm with respect to p uniformly in x at $((\bar{x}, \bar{p}), 0)$, so by Theorem 3.1 we have the conclusion. \square

4 Applications to optimization

We intend to use at-point regularity in solid set-valued optimization problems. We mention that the incompatibility between efficiency and around-point regularity generates necessary optimality conditions, as done in [17]. Once again, we are interested here to employ the weaker at-point regularity in the study of multicriteria optimization.

In this section, X, Y, Z are Banach spaces and K, Q are closed convex pointed cones in Y and Z , respectively. As usual, the cone K is proper and gives a reflexive preorder structure on Y by the equivalence $y_1 \leq_K y_2$ if and only if $y_2 - y_1 \in K$. Here the fact that K is proper means that $K \neq \{0\}$ and $K \neq Y$.

Suppose, in addition, that K is solid, i.e. its topological interior is not empty ($\text{int } K \neq \emptyset$).

Then the notion of weak efficiency with respect to the order given by K which we work with is the following.

Definition 4.1 *Let $A \subset Y$ be a nonempty subset of Y . A point $\bar{y} \in A$ is said to be a weak Pareto minimum point of A with respect to K (we write $\bar{y} \in \text{WMin}(A, K)$) if*

$$(A - \bar{y}) \cap (-\text{int } K) = \emptyset.$$

A well known feature of solid optimization is that one can use a scalarization procedure in order to look at a Pareto minimum as a minimum of a scalar problem (see, for instance, [22]). More precisely, the mechanism is described by the next result, where ∂ denotes the Fenchel subdifferential of a convex function and $\text{bd}(K)$ denotes the topological boundary of K .

Theorem 4.2 *For every $e \in \text{int } K$, the functional $s_e : Y \rightarrow \mathbb{R}$ given by*

$$s_e(z) = \inf\{\lambda \in \mathbb{R} \mid \lambda e \in z + K\} \quad (4.1)$$

is continuous, sublinear, strictly-int K -monotone and:

- (i) $\partial s_e(0) = \{v^* \in K^* \mid v^*(e) = 1\}$;
- (ii) *for every $u \in Y$, $\partial s_e(u) \neq \emptyset$ and*

$$\partial s_e(u) = \{v^* \in K^* \mid v^*(e) = 1, v^*(u) = s_e(u)\}. \quad (4.2)$$

Moreover, s_e is $d(e, \text{bd}(K))^{-1}$ -Lipschitz. If $A \subset Y$ is a nonempty set s.t. $A \cap (-\text{int } K) = \emptyset$, then $s_e(a) \geq 0$ for every $a \in A$.

For $e \in \text{int } K$ we shall denote $d(e, \text{bd}(K))^{-1}$ by L_e (the Lipschitz constant for s_e).

Now, consider a single-valued map $f : X \rightarrow Y$, a set-valued map $G : X \rightrightarrows Z$ and the vectorial problem

$$(P) \quad \text{minimize } f(x), \text{ subject to } x \in X, 0 \in G(x) + Q.$$

Naturally, a point $\bar{x} \in X$ is called a weak minimum point for (P) if $f(\bar{x}) \in \text{WMin}(f(G^{-1}(-Q)), K)$.

It is well-known that the epigraphical set-valued maps associated to both f and G play an important role in the study of vectorial problem (P) : see, for instance, [8], [17].

Consider the special type of epigraphical set-valued map associated to G as $\mathcal{E}_G : X \times Z \rightrightarrows Z$ given by

$$\mathcal{E}_G(x, q) := \begin{cases} G(x) + q, & \text{if } q \in Q, \\ \emptyset, & \text{otherwise.} \end{cases}$$

This epigraphical multifunction was successfully used in [16] in order to give necessary optimality conditions for set-valued optimization problems without constraints. At this point, just observe that if G is closed-graph, then the associated multifunction \mathcal{E}_G is a closed-graph multifunction too.

As always when one looks after necessary optimality conditions, one tries to get some (generalized) Lagrange multipliers in the normal form, i.e. to assure that the multiplier corresponding to the objective map is non-zero. Of course, this requires a qualification condition on the constraint system. The problem (P) was treated in [18] under metric regularity assumptions. Now we want to weaken the constraint qualification conditions and for this one needs the following result which employs a penalization technique for our problem similar to that in [35].

Theorem 4.3 *Suppose that $\bar{x} \in G^{-1}(-Q)$ is a weak Pareto minimum point for (P) . Fix and denote by \bar{q} a point in Q with $(\bar{x}, \bar{q}, 0) \in \text{Gr } \mathcal{E}_G$. If f is L -Lipschitz ($L > 0$) and \mathcal{E}_G is metrically subregular at $((\bar{x}, \bar{q}), 0)$ (with a constant smaller than $M > 0$) then, for every $e \in \text{int } K$, $(\bar{x}, \bar{q}, 0)$ is a local minimum point for the scalar function*

$$(x, q, z) \mapsto s_e \circ (f(x) - f(\bar{x})) + LL_e M \|z\|$$

under the constraint $(x, q, z) \in \text{Gr } \mathcal{E}_G$.

Proof. Since $f(\bar{x}) \in \text{WMin}(f(G^{-1}(-Q)), K)$ one has that

$$[f(G^{-1}(-Q)) - f(\bar{x})] \cap (-\text{int } K) = \emptyset.$$

Then, following the last assertion in Theorem 4.2, for every $e \in \text{int } K$ and $x \in G^{-1}(-Q)$

$$s_e(f(x) - f(\bar{x})) \geq 0. \quad (4.3)$$

Now, keeping in mind that the metrical subregularity of \mathcal{E}_G at $((\bar{x}, \bar{q}), 0)$ is equivalent to the calmness of \mathcal{E}_G^{-1} at $(0, (\bar{x}, \bar{q}))$, there exists $U \in \mathcal{V}(\bar{x})$, $W \in \mathcal{V}(\bar{q})$ and $V \in \mathcal{V}(0)$ such that for every $v \in V$

$$\mathcal{E}_G^{-1}(v) \cap (U \times W) \subset \mathcal{E}_G^{-1}(0) + M \|v\| D_{X \times Z}(0, 1). \quad (4.4)$$

Let $(x, q, z) \in \text{Gr } \mathcal{E}_G \cap (U \times W \times V)$. From (4.4), there is $(x_0, q_0) \in \mathcal{E}_G^{-1}(0)$ with

$$\|(x, q) - (x_0, q_0)\| \leq M \|z\|.$$

Note that, in particular, $0 \in G(x_0) + q \subset G(x_0) + Q$, hence x_0 is a feasible point, and

$$\|x - x_0\| \leq M \|z\|.$$

Now one can write, taking firstly into account (4.3),

$$\begin{aligned}
0 &= s_e \circ (f(\bar{x}) - f(\bar{x})) + LL_e M \|0\| \\
&\leq s_e(f(x_0) - f(\bar{x})) = s_e(f(x) - f(\bar{x})) + s_e(f(x_0) - f(x)) - s_e(f(x) - f(\bar{x})) \\
&\leq s_e(f(x) - f(\bar{x})) + L_e \|f(x) - f(x_0)\| \leq s_e(f(x) - f(\bar{x})) + L_e L \|x - x_0\| \\
&\leq s_e(f(x) - f(\bar{x})) + L_e LM \|z\|.
\end{aligned}$$

This proves the assertion in conclusion. \square

In order to give dual conditions for metric subregularity of \mathcal{E}_G we need to recall the mechanism of error bounds of a system given by inequality constraints.

Let X be a normed vector space and let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a given function. We set

$$S := \{x \in X : f(x) \leq 0\}. \quad (4.5)$$

One denotes the quantity $\max\{f(x), 0\}$ by $[f(x)]_+$. We say that the system (4.5) admits an error bound if there exists a real $c > 0$ such that

$$d(x, S) \leq c[f(x)]_+ \quad \text{for all } x \in X. \quad (4.6)$$

For $x_0 \in S$, we say that the system (4.5) has an error bound at x_0 , if there exists a real $c > 0$ such that relation (4.6) is satisfied for all x around x_0 .

Here and in what follows the convention $0 \cdot (+\infty) = 0$ is used.

The following result will be useful in the sequel. Notice that the parametric case is studied in [27].

Theorem 4.4 ([29, Corollary 2.3]) *Let X be a Banach space, and let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Let $\bar{x} \in S$, $\tau \in (0, +\infty)$ and $\eta \in (0, +\infty)$ be given. Consider the following statements:*

- (i) $d(x, S) \leq \tau[f(x)]_+$, for all $x \in B(\bar{x}, \eta/2)$.
- (ii) For each $x \in B(\bar{x}, \eta) \setminus S$ and for any $\varepsilon > 0$, there exists $z \in X$ such that

$$0 < d(x, z) < (\tau + \varepsilon)(f(x) - [f(z)]_+). \quad (4.7)$$

- (iii) For each $x \in B(\bar{x}, \eta) \setminus S$ and for any $\varepsilon > 0$, there exists $z \in X$ with $f(z) \geq 0$ such that (4.7) holds.
- (iv) For each $x \in B(\bar{x}, \eta) \setminus S$ and for any $\varepsilon > 0$, there exists $z \in X$ with $f(z) > 0$ such that (4.7) holds.

Then, one has $(iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$. Conversely, if (i) holds, then (ii) holds with $\eta/2$ instead of η . In addition, if f is a continuous function, then the three statements (ii), (iii), and (iv) are equivalent.

The strong slope $|\nabla f|(x)$ of a lower semicontinuous function f at $x \in \text{dom } f := \{u \in X : f(u) < +\infty\}$ is the quantity defined by $|\nabla f|(x) = 0$ if x is a local minimum of f , and by

$$|\nabla f|(x) = \limsup_{y \rightarrow x, y \neq x} \frac{f(x) - f(y)}{d(x, y)},$$

otherwise. For $x \notin \text{dom } f$, we set $|\nabla f|(x) = +\infty$ (see [10], [7]). From Theorem 4.4 we get the following result. In order to clarify the ideas, we present its proof.

Corollary 4.5 *Let X be a Banach space and let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Let $\gamma \in (0, +\infty)$ and $\bar{x} \in S$ be given. If there exist a neighborhood V of \bar{x} and a real $m > 0$ such that $|\nabla f|(x) \geq m$ for all $x \in V$ with $f(x) \in (0, \gamma)$ then there exists a neighborhood V_1 of \bar{x} such that*

$$md(x, S) \leq [f(x)]_+ \quad \text{for all } x \in V_1. \quad (4.8)$$

Proof. Without loosing the generality, suppose that $V = B(\bar{x}, \eta)$ with $\eta > 0$. Observe first that if $x \in V$ is such that $f(x) \leq 0$, then (4.8) trivially holds.

Fix now $x \in V$ such that $f(x) \in (0, \gamma)$. But this imply, on one hand, that $|\nabla f|(x) < +\infty$, and on the other hand, that there is $\alpha > 0$ such that $f(x) > \alpha > 0$. Using now the lower semicontinuity of f , we can find a neighborhood U of x such that, for every $u \in U$, $f(u) > \alpha > 0$.

Using now the assumptions of the corollary, we also know that $|\nabla f|(x) \geq m > 0$, so x cannot be a local minimum of the function f . In conclusion, by the definition of the strong slope, for every $\varepsilon > 0$, and for every $N \in \mathcal{V}(x)$, there exists $z \in N$, $z \neq x$ such that

$$(m - \varepsilon)^{-1}(f(x) - f(z)) \geq d(z, x) > 0. \quad (4.9)$$

By taking N sufficiently small, such that $N \subset U$, we obtain that $0 < f(z) = [f(z)]_+$. In conclusion, for every $x \in B(\bar{x}, \eta) \setminus S$ and every $\varepsilon > 0$, we have found $z \in X$ with $f(z) > 0$ such that

$$(m - \varepsilon)^{-1}(f(x) - [f(z)]_+) \geq d(z, x) > 0.$$

The conclusion now follows from the implication $(iv) \Rightarrow (i)$ of the previous theorem. \square

Coming back to our epigraphical set-valued map \mathcal{E}_G , define the lower semicontinuous application $\varphi_{\mathcal{E}_G} : X \times Z \times Z \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$\varphi_{\mathcal{E}_G}((x, q), z) := \liminf_{(u, r, v) \rightarrow (x, q, z)} d(v, \mathcal{E}_G(u, r))$$

which, as shown in [16], takes the form

$$\varphi_{\mathcal{E}_G}((x, q), z) = \begin{cases} \liminf_{u \rightarrow x} d(z, G(u) + q), & \text{if } q \in Q, \\ +\infty, & \text{otherwise.} \end{cases}$$

Note that, if G is a closed-graph multifunction, then for every $z \in Z$,

$$\{(x, q) \in X \times Z : y \in G(x) + q, q \in Q\} = \{(x, q) \in X \times Z : \varphi_{\mathcal{E}_G}((x, q), z) = 0\}.$$

Theorem 4.6 *Let X be a Banach space, let Z be a normed vector space and let $G : X \rightrightarrows Z$ be a closed-graph multifunction. Suppose that $(\bar{x}, \bar{q}, \bar{z}) \in X \times Z \times Z$ is such that $\bar{z} \in G(\bar{x}) + \bar{q}$ and $\bar{q} \in Q$. Let $m > 0$ be given. If there exist a neighborhood $U \times V$ of (\bar{x}, \bar{q}) and a real $\gamma > 0$ such that*

$$|\nabla \varphi_{\mathcal{E}_G}((\cdot, \cdot), \bar{z})|(x, q) \geq m \quad \text{for all } (x, q) \in U \times V \text{ with } \varphi_{\mathcal{E}_G}((x, q), \bar{z}) \in (0, \gamma), \quad (4.10)$$

then there exists a neighborhood $\tilde{U} \times \tilde{V}$ of (\bar{x}, \bar{q}) such that

$$d((x, q), \mathcal{E}_G^{-1}(\bar{z})) \leq \frac{1}{m} d(\bar{z}, G(x) + q) \quad \forall (x, q) \in \tilde{U} \times [\tilde{V} \cap Q]. \quad (4.11)$$

In other words, \mathcal{E}_G is metrically subregular at $((\bar{x}, \bar{q}), \bar{z})$, hence \mathcal{E}_G is linearly pseudo-open at $((\bar{x}, \bar{q}), \bar{z})$.

Proof. We apply Corollary 4.5 for the function

$$(x, q) \mapsto \varphi_{\mathcal{E}_G}((x, q), \bar{z}).$$

Note that in this case

$$\begin{aligned} S &= \{(x, q) \in X \times Z : \varphi_{\mathcal{E}_G}((x, q), \bar{z}) = 0\} \\ &= \{(x, q) \in X \times Z : \bar{z} \in G(x) + q, q \in Q\} \\ &= \mathcal{E}_G^{-1}(\bar{z}) \end{aligned}$$

and obviously $(\bar{x}, \bar{q}) \in S$. Since the assumptions of Corollary 4.5 are fulfilled, we can find a neighborhood $\tilde{U} \times \tilde{V}$ of (\bar{x}, \bar{q}) such that

$$md((x, q), S) \leq [\varphi_{\mathcal{E}_G}((x, q), \bar{z})]_+ = \varphi_{\mathcal{E}_G}((x, q), \bar{z}), \quad \text{for all } (x, q) \in \tilde{U} \times [\tilde{V} \cap Q].$$

Since $\varphi_{\mathcal{E}_G}((x, q), \bar{z}) \leq d(\bar{z}, G(x) + q)$ the conclusion follows. \square

The main tools for our study are the Mordukhovich's generalized differentiation objects which are the basis of many effective techniques in variational analysis. We recall the most important facts we need in this paper.

Let X be a normed vector space, S be a non-empty subset of X and let $x \in S$. The Fréchet normal cone to S at x is

$$\hat{N}(S, x) := \left\{ x^* \in X^* \mid \limsup_{u \xrightarrow{S} x} \frac{x^*(u - x)}{\|u - x\|} \leq 0 \right\}. \quad (4.12)$$

If X is an Asplund space (i.e. a Banach space where every convex continuous function is generically Fréchet differentiable), the basic (or limiting, or Mordukhovich) normal cone to S at \bar{x} is:

$$N(S, \bar{x}) = \{x^* \in X^* \mid \exists x_n \xrightarrow{S} \bar{x}, x_n^* \xrightarrow{w^*} x^*, x_n^* \in \hat{N}(S, x_n), \forall n \in \mathbb{N}\}.$$

Let $f : X \rightarrow \overline{\mathbb{R}}$, finite at $\bar{x} \in X$; the Fréchet subdifferential of f at \bar{x} is the set

$$\hat{\partial}f(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in \hat{N}(\text{epi } f, (\bar{x}, f(\bar{x})))\}$$

and, if X is Asplund, the basic (or limiting, or Mordukhovich) subdifferential of f at \bar{x} is

$$\partial f(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in N(\text{epi } f, (\bar{x}, f(\bar{x})))\},$$

where $\text{epi } f$ denotes the epigraph of f . On Asplund spaces one has

$$\partial f(\bar{x}) = \limsup_{x \xrightarrow{f} \bar{x}} \hat{\partial}f(x),$$

and, in particular, $\hat{\partial}f(\bar{x}) \subset \partial f(\bar{x})$. Note that a generalized Fermat rule holds: if \bar{x} is a local minimum point for f then $0 \in \hat{\partial}f(\bar{x})$. If f is convex, then both these subdifferentials do coincide with the classical Fenchel subdifferential. If δ_Ω denotes the indicator function associated with a nonempty set $\Omega \subset X$ (i.e. $\delta_\Omega(x) = 0$ if $x \in \Omega$, $\delta_\Omega(x) = \infty$ if $x \notin \Omega$), then for any $\bar{x} \in \Omega$,

$\widehat{\partial}\delta_\Omega(\bar{x}) = \widehat{N}(\Omega, \bar{x})$ and $\partial\delta_\Omega(\bar{x}) = N(\Omega, \bar{x})$. Let $\Omega \subset X$ be a nonempty set and take $\bar{x} \in \Omega$; then one has:

$$\widehat{\partial}d(\cdot, \Omega)(\bar{x}) = \widehat{N}(\Omega, \bar{x}) \cap D_{X^*}, \quad \widehat{N}(\Omega, \bar{x}) = \bigcup_{\lambda > 0} \lambda \widehat{\partial}d(\cdot, \Omega)(\bar{x}). \quad (4.13)$$

The basic subdifferential satisfies a robust sum rule (see [26, Theorem 3.36]): if X is Asplund, f_1 is Lipschitz around \bar{x} and f_2 is lower semicontinuous around this point, then

$$\partial(f_1 + f_2)(\bar{x}) \subset \partial f_1(\bar{x}) + \partial f_2(\bar{x}). \quad (4.14)$$

A function $f : X \rightarrow Y$ is said to be strictly Lipschitz at \bar{x} if it is locally Lipschitzian around this point and there exists a neighborhood V of the origin in X s.t. the sequence $(t_k^{-1}(f(x_k + t_k v) - f(x_k)))_{k \in \mathbb{N}}$ contains a norm convergent subsequence whenever $v \in V, x_k \rightarrow \bar{x}, t_k \downarrow 0$.

Suppose that X, Y are Asplund spaces. Let $f : X \rightarrow Y$ and $\varphi : Y \rightarrow \mathbb{R}$ s.t. f is strictly Lipschitz at $\bar{x} \in X$ and φ is Lipschitz around $f(\bar{x})$; then

$$\partial(\varphi \circ f)(\bar{x}) \subset \bigcup_{y^* \in \partial\varphi(f(\bar{x}))} \partial(y^* \circ f)(\bar{x}). \quad (4.15)$$

Let $F : X \rightrightarrows Y$ be a set-valued map and $(\bar{x}, \bar{y}) \in \text{Gr } F$. Then the Fréchet coderivative at (\bar{x}, \bar{y}) is the set-valued map $\widehat{D}^*F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ given by

$$\widehat{D}^*F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in \widehat{N}(\text{Gr } F, (\bar{x}, \bar{y}))\}.$$

Similarly, on Asplund spaces, the normal coderivative of F at (\bar{x}, \bar{y}) is the set-valued map $D^*F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ given by

$$D^*F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N(\text{Gr } F, (\bar{x}, \bar{y}))\}.$$

Now, the next theorem is a reformulation of the main result in [16], where by Q^* we denote the positive polar cone of Q .

Theorem 4.7 *Let X, Z be Banach spaces and G be closed-graph. Then for each $(x, q_0, z) \in X \times Q \times Z$ with $z \notin G(x) + q_0$, one has*

$$|\nabla \varphi_{\mathcal{E}_G}((\cdot, \cdot), z)|(x, q_0) \geq \lim_{\rho \downarrow 0} \left\{ \inf \left\{ \|x^*\| : \begin{array}{l} (u, v) \in \text{Gr } G, \ u \in B(x, \rho), \ x^* \in \widehat{D}^*G(u, v)(z^* + v^*), \\ z^* \in Q^* \cap S_{Z^*}, \ v^* \in \rho B_{Z^*}, \\ d(z, G(u) + q_0) \leq \varphi_{\mathcal{E}_G}((x, q_0), z) + \rho, \\ \|z - v - q_0\| \leq d(z, G(u) + q_0) + \rho, \\ |\langle z^* + v^*, z - v - q_0 \rangle - d(z, G(u) + q_0)| < \rho \end{array} \right\} \right\}. \quad (4.16)$$

Now, putting together Theorems 4.6 and 4.7 one gets a metric subregularity sufficient condition in terms of dual objects.

Theorem 4.8 Let X, Z be Banach spaces and G be closed-graph. Suppose that $(\bar{x}, \bar{q}, \bar{z}) \in X \times Z \times Z$ is such that $\bar{z} \in G(\bar{x}) + \bar{q}$ and $\bar{q} \in Q$. Let $m > 0$ be given. If there exist a neighborhood $U \times V$ of (\bar{x}, \bar{q}) and a real $\gamma > 0$ such that for every $(x, q) \in U \times [V \cap Q]$ with $\bar{z} \notin G(x) + q$,

$$m \leq \lim_{\rho \downarrow 0} \left\{ \inf \left\{ \|x^*\| : \begin{array}{l} (u, v) \in \text{Gr } G, \ u \in B(x, \rho), \ x^* \in \widehat{D}^*G(u, v)(z^* + v^*), \\ z^* \in Q^* \cap S_{Z^*}, \ v^* \in \rho B_{Z^*}, \\ d(\bar{z}, G(u) + q) \leq \gamma + \rho, \\ \|\bar{z} - v - q\| \leq d(\bar{z}, G(u) + q) + \rho, \\ |\langle z^* + v^*, \bar{z} - v - q \rangle - d(\bar{z}, G(u) + q)| < \rho \end{array} \right\} \right\} \quad (4.17)$$

then \mathcal{E}_G is metrically subregular at $((\bar{x}, \bar{q}), \bar{z})$.

Proof. Take arbitrary $(x, q) \in U \times V$ with $\varphi_{\mathcal{E}_G}((x, q), \bar{z}) \in (0, \gamma)$. Then $q \in Q$, because otherwise $\varphi_{\mathcal{E}_G}((x, q), \bar{z}) = +\infty$. Also, $\bar{z} \notin G(x) + q$, because otherwise $\varphi_{\mathcal{E}_G}((x, q), \bar{z}) = 0$. Now, since the set from (4.16) (where the infimum is taken) is smaller than the corresponding set from (4.17), we get that $|\nabla \varphi_{\mathcal{E}_G}((\cdot, \cdot), \bar{z})|(x, q) \geq m$. But this means, in virtue of Theorem 4.6, exactly the conclusion. \square

In some cases, the condition (4.17) could be simplified, as the next corollary shows.

Corollary 4.9 Let X, Z be Asplund spaces and G be closed-graph. Suppose that $(\bar{x}, \bar{z}) \in X \times Z$ is such that $\bar{z} \in G(\bar{x})$. If there exist $r, c > 0$ such that, for every $(x, q, z) \in B(\bar{x}, r) \times B(0, r) \times B(\bar{z}, r)$, $(x, z) \in \text{Gr } G$, $(x, q, \bar{z}) \notin \text{Gr } \mathcal{E}_G$, $z^* \in Q^* \cap S_{Z^*}$, $v^* \in 2cB_{Y^*}$, $x^* \in D^*G(x, z)(z^* + v^*)$,

$$c\|z^* + v^*\| \leq \|x^*\|,$$

then \mathcal{E}_G linearly pseudo-open at $((\bar{x}, 0), \bar{z})$.

Proof. Fix $m := 2^{-1}c > 0$, $\gamma := 2^{-1}r > 0$, $U := B(\bar{x}, 2^{-1}r)$, $V := B(0, 4^{-1}r)$.

Suppose first that for arbitrary $(x, q) \in U \times [V \cap Q]$, one has that $\bar{z} \in G(x) + q = \mathcal{E}_G(x, q)$. It follows in particular that for every $(x, q) \in B(\bar{x}, 2^{-1}r) \times [B(0, r) \cap Q]$, one has that $\bar{z} \in \mathcal{E}_G(x, q)$. Consequently, for every $\tau \in (0, 2^{-1}r)$, one has that

$$\bar{z} \in \mathcal{E}_G(x, q) \subset \mathcal{E}_G(B(x, \tau), B(q, \tau)).$$

As the previous relation is true for every $(x, q) \in U \times [V \cap Q]$ and every $\tau \in (0, 2^{-1}r)$, we conclude that \mathcal{E}_G is linearly pseudo-open at $((\bar{x}, 0), \bar{z})$.

Suppose now that there exists $(x, q) \in U \times [V \cap Q]$ such that $\bar{z} \notin G(x) + q$, i.e. $(x, q, \bar{z}) \notin \text{Gr } \mathcal{E}_G$. Choose $\rho \in (0, \min\{8^{-1}r, 2c, 2^{-1}\})$. Consider $(u, v) \in \text{Gr } G$, $u \in B(x, \rho)$, $z^* \in Q^* \cap S_{Z^*}$, $v^* \in \rho B_{Z^*}$, $x^* \in \widehat{D}^*G(u, v)(z^* + v^*)$, $\|\bar{z} - v - q\| \leq d(\bar{z}, G(u) + q) + \rho$, $d(\bar{z}, G(u) + q) \leq \gamma + \rho$, $|\langle z^* + v^*, \bar{z} - v - q \rangle - d(\bar{z}, G(u) + q)| < \rho$. Then

$$\begin{aligned} \|u - \bar{x}\| &\leq \|u - x\| + \|x - \bar{x}\| < \rho + 2^{-1}r < r, \\ \|v - \bar{z}\| &\leq \|v + q - \bar{z}\| + \|q\| < d(\bar{z}, G(u) + q) + \rho + 4^{-1}r \\ &\leq \gamma + 2\rho + 4^{-1}r < 2^{-1}r + 4^{-1}r + 4^{-1}r = r, \\ \|v^*\| &< \rho < 2c. \end{aligned}$$

One can use now the hypothesis from the statement of the Corollary to get that

$$\|x^*\| \geq c\|z^* + v^*\| \geq c(\|z^*\| - \|v^*\|) \geq c(1 - \rho) \geq m.$$

Using now Theorem 4.8, one gets the conclusion. \square

Summing up, Theorem 4.3 gives a scalarization method for (P) under at-point assumption of the epigraphical multifunction associated to the constraints, while Theorem 4.6 provides sufficient conditions for the fulfilment of this assumption. Finally, taking advantage of the power of Mordukhovich subdifferential calculus we present necessary optimality conditions for (P) in terms of generalized differentiation.

Theorem 4.10 *Take X, Y, Z as Asplund spaces and $\bar{x} \in G^{-1}(-Q)$ as a weak Pareto minimum point for (P) . Fix $\bar{q} \in Q$ such that $(\bar{x}, \bar{q}, 0) \in \text{Gr } \mathcal{E}_G$. Suppose that*

- (i) f is L -Lipschitz ($L > 0$) and strictly Lipschitz at \bar{x} ;*
 - (ii) G is closed-graph;*
 - (iii) \mathcal{E}_G is metrically subregular at $((\bar{x}, \bar{q}), 0)$ (with a constant smaller than $M > 0$).*
- Then, for every $e \in \text{int } K$, there exist $y^* \in K^*$, $y^*(e) = 1$, $z^* \in Z^*$, $\|z^*\| \leq LL_e M$ s.t.*

$$(0, 0) \in \partial(y^* \circ f)(\bar{x}) \times \{0\} + D^* \mathcal{E}_G(\bar{x}, \bar{q}, 0)(z^*).$$

Proof. Consider $e \in \text{int } K$. Taking into account Theorem 4.3, $(\bar{x}, \bar{q}, 0)$ is a local minimum point for the scalar function

$$(x, q, z) \mapsto s_e \circ (f(x) - f(\bar{x})) + LL_e M \|z\|$$

under the constraint $(x, q, z) \in \text{Gr } \mathcal{E}_G$. As usual, this means that $(\bar{x}, \bar{q}, 0)$ is a local minimum point for the unconstrained scalar problem

$$\min [s_e \circ (f(\cdot) - f(\bar{x})) + LL_e M \|\cdot\| + \delta_{\text{Gr } \mathcal{E}_G}(\cdot, \cdot, \cdot)].$$

Consequently,

$$(0, 0, 0) \in \partial [s_e \circ (f(\cdot) - f(\bar{x})) + LL_e M \|\cdot\| + \delta_{\text{Gr } \mathcal{E}_G}(\cdot, \cdot, \cdot)](\bar{x}, \bar{q}, 0).$$

Since under our assumption we can apply the exact sum rule (4.14) we get (by means of some obvious calculations):

$$(0, 0, 0) \in \partial s_e \circ (f(\cdot) - f(\bar{x}))(\bar{x}) \times \{0\} \times \{0\} + LL_e M [\{0\} \times \{0\} \times D_{Z^*}(0, 1)] + \partial \delta_{\text{Gr } \mathcal{E}_G}(\bar{x}, \bar{q}, 0).$$

The hypothesis (i) gives us the right to use the chain rule (4.15): there exists $y^* \in \partial s_e(0)$ s.t.

$$(0, 0, 0) \in \partial(y^* \circ f)(\bar{x}) \times \{0\} \times \{0\} + LL_e M [\{0\} \times \{0\} \times D_{Z^*}(0, 1)] + N(\text{Gr } \mathcal{E}_G, (\bar{x}, \bar{q}, 0)).$$

Then there exist $u^* \in \partial(y^* \circ f)(\bar{x})$, $z^* \in D_{Z^*}(0, LL_e M)$ with

$$(-u^*, 0, -z^*) \in N(\text{Gr } \mathcal{E}_G, (\bar{x}, \bar{q}, 0)),$$

i.e.

$$(-u^*, 0) \in D^* \mathcal{E}_G(\bar{x}, \bar{q}, 0)(z^*)$$

which achieves the proof. \square

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